A taste of statistics and applications to X-ray spectral fitting

- ✓ Normal error (Gaussian) distribution
 - → most important in statistical analysis of data, describes the distribution of random observations for many experiments
- ✓ Poisson distribution
 - → generally appropriate for counting experiments related to random processes (e.g., radioactive decay of elementary particles)
- ✓ Statistical tests: χ^2 and F-test
- ✓ Statistical errors and contour plots
- ✓ Low-count regime: the C-statistic

✓ Applications to X-ray spectral fitting

The Gaussian (normal error) distribution

Casual errors are above and below the "true" (most "common") value → bell-shape distribution if systematic errors are negligible



Gaussian probability function

$$P(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

normalization factor, so that $\int f(x) dx = 1$

Probability Density Function (centered on μ)

μ=mean valueσ=standard deviation

$$e^{-x^2/2\sigma^2}$$

function centered on 0



The Poisson distribution

Describes experimental results where events are counted and the uncertainty is not related to the measurement but reflects the intrinsically casual behavior of the process (e.g., radioactive decay of particles, X-ray photons, etc.)

$$P(x) = e^{-\mu} \mu^x / x! \quad (x=0,1,2,...)$$

Probability of obtaining x events when the expected number is μ μ >0: main parameter of the Poisson distribution x=observed number of events in a time interval (frequency of events)

average number of events

$$\overline{x} = \sum_{x=0}^{\infty} x P(x) = \sum_{x=0}^{\infty} x e^{-\mu} \mu^{x} / x! = \mu$$

 \rightarrow µ=average number of expected events if the experiment is repeated many times

$$\sigma^2 = \left\langle (x - \mu)^2 \right\rangle = \cdots$$
$$= \sum_{x=0}^{\infty} (x - \mu)^2 \frac{\mu^x}{x!} e^{-\mu} = \mu$$

expectation value of the square of the deviations

the Poisson distribution with average counts= μ has standard deviation $\sqrt{\mu}$

μ » : the Poisson distribution is approximated by the Gaussian distribution

defined by only one parameter μ



Statistical tests: χ^2

Test to compare the observed distribution of the results with that expected

$$\chi^2 = \sum_{k=1}^n \frac{(O_k - E_k)}{\sigma_k^2}^2$$

 O_k =observed values E_k =expected values σ_k =error on the measured values k=number of intervals

$$\chi^2 \leq n$$

the observed and expected distributions are similar

Degrees of freedom

Degrees of freedom (d.o.f.)=number of observed data – number of parameters computed from the data and used in the calculation

d.o.f.=n-c

where

n=number of data (e.g., spectral bins)

c=number of parameters which must be computed from the data to obtain the expected E_k



X-ray spectral fits

d.o.f.=number of spectral data points – number of free parameters





TABLE C.4 χ^2 distribution. Values of the reduced chi-square $\chi^2_{\nu} = \chi^2 / \nu$ corresponding to the probability $P_{\chi}(\chi^2; \nu)$ of exceeding χ^2 vs. the number of degrees of freedom ν

	Р													
v	0.99	0.98	10.95	0.90	0.80	0.70	0.60	0.50						
1	0.00016	0.00063	0.00393	0.0158	0.0642	0.148	0.275	0.455						
2	0.0100	0.0202	0.0515	0.105	0.223	0.357	0.511	0.693						
3	0.0383	0.0617	0.117	0.195	0.335	0.475	0.623	0.789						
4	0.0742	0.107	0.178	0.266	0.412	0.549	0.688	0.839						
5	0.111	0.150	0.229	0.322	0.469	0.600	0.731	0.870						
6	0.145	0.189	0.273	0.367	0.512	0.638	0.762	0.891						
7	0.177	0.223	0.310	0.405	0.546	0.667	0.785	0.907						
8	0.206	0.254	0.342	0.436	0.574	0.691	0.803	0.918						
9	0.232	0.281	0.369	0.463	0.598	0.710	0.817	0.927						
10	0.256	0.306	0.394	0.487	0.618	0.727	0.830	0.934						
11	0.278	0.328	0.416	0.507	0.635	0.741	0.840	0.940						
12	0.298	0.348	0.436	0.525	0.651	0.753	0.848	0.945						
13	0.316	0.367	0.453	0.542	0.664	0.764	0.856	0.949						
14	0.333	0.383	0.469	0.556	0.676	0.773	0.863	0.953						
15	0.349	0.399	0.484	0.570	0.687	0.781	0.869	0.956						
16	0.363	0.413	0.498	0.582	0.697	0.789	0.874	0.959						
17	0.377	0.427	0.510	0.593	0.706	0.796	0.879	0.961						
18	0.390	0.439	0.522	0.604	0.714	0.802	0.883	0.963						
19	0.402	0.451	0.532	0.613	0.722	0.808	0.887	0.965						
20	0.413	0.462	0.543	0.622	0.729	0.813	0.890	0.967						
22	0.434	0.482	0.561	0.638	0.742	0.823	0.897	0.970						
24	0.452	0.500	0.577	0.652	0.753	0.831	0.902	0.972						
26	0.469	0.516	0.592	0.665	0.762	0.838	0.907	0.974						
28	0.484	0.530	0.605	0.676	0.771	0.845	0.911	0.976						
30	0.498	0.544	0.616	0.687	0.779	0.850	0.915	0.978						
32	0.511	0.556	0.627	0.696	0.786	0.855	0.918	0.979						
34	0.523	0.567	0.637	0.704	0.792	0.860	0.921	0.980						
36	0.534	0.577	0.646	0.712	0.798	0.864	0.924	0.982						
38	0.545	0.587	0.655	0.720	0.804	0.868	0.926	0.983						
40	0.554	0.596	0.663	0.726	0.809	0.872	0.928	0.983						
42	0.563	0.604	0.670	0.733	0.813	0.875	0.930	0.984						
44	0.572	0.612	0.677	0.738	0.818	0.878	0.932	0.985						
46	0.580	0.620	0.683	0.744	0.822	0.881	0.934	0.986						
48	0.587	0.627	0.690	0.749	0.825	0.884	0.936	0.986						
50	0.594	0.633	0.695	0.754	0.829	0.886	0.937	0.987						
60	0.625	0.662	0.720	0.774	0.844	0.897	0.944	0.989						
70	0.649	0.684	0.739	0.790	0.856	0.905	0.949	0.990						
80	0.669	0.703	0.755	0.803	0.865	0.911	0.952	0.992						
90	0.686	0.718	0.768	0.814	0.873	0.917	0.955	0.993						
100	0.701	0.731	0.779	0.824	0.879	0.921	0.958	0.993						
120	0.724	0.753	0.798	0.839	0.890	0.928	0.962	0.994						
140	0.743	0.770	0.812	0.850	0.898	0.934	0.965	0.995						
160	0.758	0.784	0.823	0.860	0.905	0.938	0.968	0.996						
180	0.771	0.796	0.833	0.868	0.910	0.942	0.970	0.996						
200	0.782	0.806	0.841	0.874	0.915	0.945	0.972	0.997						

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v	P 0.40 0.30 0.20 0.10 0.05 0.02 0.01											
	0.40	0.50	0.20	0.10	0.05	0.02	0.01	0.001				
1	0.708	1.074	1.642	2.706	3.841	5.412	6.635	10.82				
2	0.916	1.204	1.609	2.303	2.996	3.912	4.605	6.90				
3	0.982	1.222	1.547	2.084	2.605	3.279	3.780	5.42				
4	1.011	1.220	1.497	1.945	2.372	2.917	3.319	4.61				
2	1.026	1.213	1.458	1.847	2.214	2.678	3.017	4.10				
6	1.035	1.205	1.426	1.774	2 000	2 506	2 002					
7	1.040	1.198	1.400	1.717	2.010	2.300	2.602	3.74.				
8	1.044	1.191	1.379	1.670	1 038	2.373	2.039	3.47				
9	1.046	1.184	1.360	1.632	1.880	2.271	2.511	3.260				
10	1.047	1.178	1.344	1 599	1.831	2.10/	2.407	3.09				
11	1 049	1 1 7 7		1.577	1.051	2.110	2.321	2.959				
12	1.048	1.173	1.330	1.570	1.789	2.056	2.248	2.842				
12	1.049	1.168	1.318	1.546	1.752	2.004	2.185	2.742				
13	1.049	1.163	1.307	1.524	1.720	1.959	2.130	2.656				
14	1.049	1.159	1.296	1.505	1.692	1.919	2.082	2.580				
15	1.049	1.155	1.287	1.487	1.666	1.884	2.039	2.513				
16	1.049	1.151	1.279	1.471	1.644	1.852	2 000	2 457				
17	1.048	1.148	1.271	1.457	1.623	1.823	1 965	2.453				
18	1.048	1.145	1.264	1.444	1.604	1 797	1.905	2.399				
19	1.048	1.142	1.258	1.432	1.586	1 773	1.904	2.331				
20	1.048	1.139	1.252	1.421	1.571	1 751	1.903	2.307				
22	1.047	1 134	1.241	1 101		1.751	1.070	2.200				
24	1.047	1.134	1.241	1.401	1.542	1.712	1.831	2.194				
26	1.045	1.129	1.231	1.383	1.517	1.678	1.791	2.132				
28	1.045	1.125	1.223	1.368	1.496	1.648	1.755	2.079				
30	1.045	1.121	1.215	1.354	1.476	1.622	1.724	2.032				
50	1.044	1.110	1.208	1.342	1.459	1.599	1.696	1.990				
32	1.043	1.115	1.202	1.331	1.444	1.578	1 671	1 053				
34	1.042	1.112	1.196	1.321	1.429	1.559	1.649	1.933				
36	1.042	1.109	1.191	1.311	1.417	1.541	1.628	1 999				
38	1.041	1.106	- 1.186	1.303	1.405	1.525	1.610	1 861				
40	1.041	1.104	1.182	1.295	1.394	1.511	1.592	1.835				
42	1.040	1.102	1 178	1 288	1 204	1.407		1.033				
44	1.039	1.100	1 174	1.200	1.384	1.497	1.576	1.812				
46	1.039	1.098	1.174	1.201	1.375	1.485	1.562	1.790				
48	1.038	1.096	1.170	1.2/5	1.300	1.473	1.548	1.770				
50	1.038	1.090	1.107	1.269	1.358	1.462	1.535	1.751				
		1.094	1.103	1.263	1.350	1.452	1.523	1.733				
60	1.036	1.087	1.150	1.240	1.318	1.410	1.473	1.660				
/0	1.034	1.081	1.139	1.222	1.293	1.377	1.435	1 605				
80	1.032	1.076	1.130	1.207	1.273	1.351	1.404	1.560				
90	1.031	1.072	1.123	1.195	1.257	1.329	1.379	1.500				
.00	1.029	1.069	1.117	1.185	1.243	1.311	1.358	1 404				
20	1.027	1.063	1.107	1 160	1 221	1 202	1.550	1.494				
40	1.026	1.059	1.099	1.109	1.221	1.283	1.325	1.446				
60	1.024	1.055	1.093	1.130	1.204	1.261	1.299	1.410				
80	1.023	1.052	1.095	1.140	1.191	1.243	1.278	1.381				
00	1 022	1.050	1.007	1.157	1.179	1.228	1.261	1.358				

χ² test in a nutshell

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Reduced \chi^2 large \bigstar P(\chi^2) small:
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(a)errors are under-estimated(b)F(x) does not describe the data very well (bad fit)

Reduced χ^2 small $\leftarrow \rightarrow P(\chi^2)$ large:

(a)errors are over-estimated(b)data specially selected?

Real case when limited photon statistics applies (too many channels with 0–1 counts)

F-test

If two statistics following the χ^2 distribution have been determined, the ratio of the reduced chi-squares is distributed according to the F distribution

$$P_f(f;v_1,v_2) = \frac{\chi_1^2 / v_1}{\chi_2^2 / v_2}$$

$$\propto \Delta \chi^2 / k$$

with k=number of additional terms (parameters)

Example: Use the F-test to evaluate the improvement to a spectral fit due to the assumption of a different model, with additional terms
Conditions: (a) the simpler model is nested within the more complex model;
(b) the extra parameters have Gaussian distribution (not truncated by the parameter space boundaries)
→see the F-test tables for the corresponding probabilities (specific command in XSPEC)

An application of the F-test







$$F_t = \left(\frac{\chi^2(dof) - \chi^2(dof - k)}{dof - (dof - k)}\right) / (\chi^2(dof - k)/(dof - k)) =$$

 $=(\Delta \chi^2/k)/\chi^2_{\nu}$

Ex: $\chi^2(103) = 97.23$ $\chi^2(101) = 90.84$ $\rightarrow \Delta \chi^2 = 6.39, k = 2 \rightarrow F_t = (6.39/2)/(90.84/101) = 3.55$

 F_t follows the F distribution with $v_1 = k = \Delta(dof)$ and $v_2 = dof - k(-1)$

Search in the F-distribution tables for the probability of the null hypothesis (H₀) for v₁=2 and v₂=100

v ₁ =2									TAB	$\frac{1.E}{P(F)} =$	(Con = 0.05	td.)									230	D/E		0E
$v_{0} = 100$	f2 f1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	8		P(F)=0	.05
(60-120)	1 2 3 4 5	161.45 18.51 10.13 7.71 6.61	199.30 19.00 9.55 6.94 5.79	$\begin{array}{c} 215.71 \\ 19.16 \\ 9.28 \\ 6.59 \\ 5.41 \end{array}$	$\begin{array}{r} 224.58 \\ 19.25 \\ 9.12 \\ 6.39 \\ 5.19 \end{array}$	$230.16 \ 2$ $19.30 \ 9.01 \ 6.26 \ 5.05$	$233.99 \\19.33 \\8.94 \\6.16 \\4.95$	236.77 19.35 8.89 6.09 4.88	238.88 19.37 8.85 6.04 4.82	$\begin{array}{r} 240.54 \\ 19.38 \\ 8.81 \\ 6.00 \\ 4.77 \end{array}$	$\begin{array}{r} 241.88 \\ 19.40 \\ 8.79 \\ 5.96 \\ 4.74 \end{array}$	$\begin{array}{r} 43.91 \\ 19.41 \\ 8.74 \\ 5.91 \\ 4.68 \end{array}$	$245.95 2 \\19.43 \\8.70 \\5.86 \\4.62$	$ \begin{array}{r} 248.01 \\ 19.45 \\ 8.66 \\ 5.80 \\ 4.56 \\ \end{array} $	$249.05 \\ 19.45 \\ 8.64 \\ 5.77 \\ 4.53$	250.09 2 19.46 8.62 5.75 4.50		252.20 19.48 8.57 5.69 4.43	253.25 2 19.49 8.55 5.66 4.40	254.32 19.50 8.53 5.63 4.36				
(00 120)	6 7 8 9 10	5.99 5.59 5.32 5.12 4.96	5.14 4.74 4.46 4.26 4.10	$\begin{array}{r} 4.76 \\ 4.35 \\ 4.07 \\ 3.86 \\ 3.71 \end{array}$	$\begin{array}{r} 4.53 \\ 4.12 \\ 3.84 \\ 3.63 \\ 3.48 \end{array}$	4.39 3.97 3.69 3.48 3.33	4.28 3.87 3.58 3.37 3.22	4.21 3.79 3.50 3.29 3.14	$\begin{array}{r} 4.15 \\ 3.73 \\ 3.44 \\ 3.23 \\ 3.07 \end{array}$	$\begin{array}{r} 4.10 \\ 3.68 \\ 3.39 \\ 3.18 \\ 3.02 \end{array}$	4.06 3.64 3.35 3.14 2.98	4.00 3.57 3.28 3.07 2.91	$3.94 \\ 3.51 \\ 3.22 \\ 3.01 \\ 2.84$	3.87 3.44 3.15 2.94 2.77	3.84 3.41 3.12 2.90 2.74	3.81 3.38 3.08 2.86 2.70	3.77 3.34 3.04 2.83 2.66	3.74 3.30 3.01 2.79 2.62	3.70 3.27 2.97 2.75 2.58	3.67 3.23 2.93 2.71 2.54				
F=3.15.3.07	11 12 13 14 15	$\begin{array}{r} 4.84 \\ 4.75 \\ 4.67 \\ 4.60 \\ 4.54 \end{array}$	3.98 3.89 3.81 3.74 3.68	3.59 3.49 3.41 3.34 3.29	$3.36 \\ 3.26 \\ 3.18 \\ 3.11 \\ 3.06$	3.20 3.11 3.03 2.96 2.90	3.09 3.00 2.92 2.85 2.79	3.01 2.91 2.83 2.76 2.71	2.95 2.85 2.77 2.70 2.64	2.90 2.80 2.71 2.65 2.59	2.85 2.75 2.67 2.60 2.54	2.79 2.69 2.60 2.53 2.48	2.72 2.62 2.53 2.46 2.40	2.65 2.54 2.46 2.39 2.33	2.61 2.51 2.42 2.35 2.29	2.57 2.47 2.38 2.31 2.25	2.53 2.43 2.34 2.27 2.20	2.49 2.38 2.30 2.22 2.16	2.45 2.34 2.25 2.18 2.11	2.40 2.30 2.21 2.13 2.07				
at P(F)=0.05	16 17 18 19 20	$\begin{array}{r} 4.49 \\ 4.45 \\ 4.41 \\ 4.38 \\ 4.35 \end{array}$	3.63 3.59 3.55 3.52 3.49	$3.24 \\ 3.20 \\ 3.16 \\ 3.13 \\ 3.10$	3.01 2.96 2.93 2.90 2.87	2.85 2.81 2.77 2.74 2.71	2.74 2.70 2.66 2.63 2.60	$2.66 \\ 2.61 \\ 2.58 \\ 2.54 \\ 2.51$	2.59 2.55 2.51 2.48 2.45	2.54 2.49 2.46 2.42 2.39	2.49 2.45 2.41 2.38 2.35	2.42 2.38 2.34 2.31 2.28	2.35 2.31 2.27 2.23 2.20	2.28 2.23 2.19 2.16 2.12	2.24 2.19 2.15 2.11 2.08	2.19 2.15 2.11 2.07 2.04	2.15 2.10 2.06 2.03 1.99	2.11 2.06 2.02 1.98 1.95	2.06 2.01 1.97 1.93 1.90	2.01 1.96 1.92 1.88 1.84				
	21 22 23 24 25	$\begin{array}{r} 4.32 \\ 4.30 \\ 4.28 \\ 4.26 \\ 4.24 \end{array}$	3.47 3.44 3.42 3.40 3.39	3.07 3.05 3.03 3.01 2.99	2.84 2.82 2.80 2.78 2.76	2.68 2.66 2.64 2.62 2.60	2.57 2.55 2.53 2.51 2.49	$2.49 \\ 2.46 \\ 2.44 \\ 2.42 \\ 2.40$	2.42 2.40 2.37 2.36 2.34	2.37 2.34 2.32 2.30 2.28	2.32 2.30 2.27 2.25 2.24	2.25 2.23 2.20 2.18 2.16	2.18 2.15 2.13 2.11 2.09	2.10 2.07 2.05 2.03 2.01	2.05 2.03 2.00 1.98 1.96	2.01 1.98 1.96 1.94 1.92	1.96 1.94 1.91 1.89 1.87	1.92 1.89 1.86 1.84 1.82	1.87 1.84 1.81 1.79 1.77	1.81 1.78 1.76 1.73 1.71				
F=3.93,3.80	26 27 28 29 30	$\begin{array}{r} 4.23 \\ 4.21 \\ 4.20 \\ 4.18 \\ 4.17 \end{array}$	3.37 3.35 3.34 3.33 3.32	2.98 2.96 2.95 2.93 2.92	2.74 2.73 2.71 2.70 2.69	2.59 2.57 2.56 2.55 2.53	2.47 2.46 2.45 2.43 2.42	2.39 2.37 2.36 2.35 2.33	2.32 2.31 2.29 2.28 2.27	2.27 2.25 2.24 2.22 2.21	2.22 2.20 2.19 2.18 2.16	2.15 2.13 2.12 2.10 2.09	2.07 2.06 2.04 2.03 2.01	$1.99 \\ 1.97 \\ 1.96 \\ 1.94 \\ 1.93$	1.95 1.93 1.91 1.90 1.89	1.90 1.88 1.87 1.85	1.85 1.84 1.82 1.81	1.80 1.79 1.77 1.75	1.75 1.73 1.71 1.70 1.69	1.69 1.67 1.65 1.64				
at P(F)=0.025	60 120 ∞	4.08 4.00 3.92 3.84	3.15 3.07 3.00	2.84 2.76 2.68 2.60	2.61 2.53 2.45 2.37	2.45 2.37 2.29 2.21	2.34 2.25 2.18 2.10	2.25 2.17 2.09 2.01	2.18 2.10 2.02 1.94	2.12 2.04 1.96	2.08 1.99 1.91	2.00 1.92 1.83	1.92 1.84 1.75	1.84 1.75 1.66	1.79 1.70 1.61	1.74 1.65 1.55	1.69 1.59 1.50	1.64 1.53 1.43	1.58 1.47 1.35	1.51 1.39 1.25				
			-	-				-					101	1.01	1.02	1.40	1.00	1.02	1.22	1.00				
F _{xspec} =3.55										P(F) =	0.025	1										P(F))=0.	025
F _{xspec} =3.55	f2 f1	1	2	3	4	5	6	7	8	P(F) = 9 63.28 9	0.025 10 68.63 9	12 76.71 9	15	20 93.10 9	24	30 1001.4 1	40 005.6 1	60 1009.8 1 39.48	120 014.0 1	∞ 018.3 39.50		P(F))=0.	025
F _{xspec} =3.55	$f_2 = f_1$ $f_2 = f_1$ $f_2 = f_1$ $f_2 = f_2$ $f_3 = f_1$ $f_2 = f_2$ $f_3 = f_1$ $f_2 = f_2$ $f_3 = f_2$ $f_3 = f_2$ $f_4 = f_2$ $f_5 = f_1$ $f_5 = f_2$	1 347.79 38.51 17.44 12.22 10.01	2 799.50 39.00 16.04 10.65 8.43	3 64.16 8 39.16 15.44 9.98 7.76	4 999.58 9 39.25 15.10 9.60 7.39	5 21.85 9 39.30 14.88 9.36 7.15	6 37.11 9 39.33 14.74 9.20 6.98	7 448.22 9 39.36 14.62 9.07 6.85	8 56.66 9 39.37 14.54 8.98 6.76	P(F) = 9 63.28 9 39.39 14.47 8.90 6.68	0.025 10 68.63 9 39.40 14.42 8.84 6.62	12 76.71 9 39.42 14.34 8.75 6.52	15 84.87 9 39.43 14.25 8.66 6.43	20 93.10 9 39.45 14.17 8.56 6.33	24 997.25 1 39.46 14.12 8.51 6.28	30 001.4 1 39.46 14.08 8.46 6.23	40 005.6 1 39.47 14.04 8.41 6.18	60 1009.8 1 39.48 13.99 8.36 6.12	120 014.0 1 39.49 13.95 8.31 6.07	∞ 018.3 39.50 13.90 8.26 6.02 4.85		P(F))=0.	025
F _{xspec} =3.55	f_2 f_1 f_2 f_3 f_4 f_5 f_7 f_8 g_9 f_9	1 347.79 38.51 17.44 12.22 10.01 8.81 8.07 7.57 7.21 6.94	2 799.50 8 39.00 16.04 10.65 8.43 7.26 6.54 6.54 6.06 5.71 5.46	3 64.16 8 39.16 15.44 9.98 7.76 6.60 5.89 5.42 5.08 4.83	4 999.58 9 39.25 15.10 9.60 7.39 6.23 5.52 5.05 4.72 4.47	5 21.85 9 39.30 14.88 9.36 7.15 5.99 5.29 4.82 4.82 4.24	6 337.11 9 39.33 14.74 9.20 6.98 5.82 5.12 4.65 4.32 4.07	7 48.22 39.36 14.62 9.07 6.85 5.70 4.99 4.53 4.20 3.95	$\begin{array}{c} 8\\ 56.66& 9\\ 39.37\\ 14.54\\ 8.98\\ 6.76\\ 5.60\\ 4.90\\ 4.43\\ 4.10\\ 3.85\end{array}$	P(F) = 9 9 $63.28 9$ 39.39 14.47 8.90 6.68 5.52 4.82 4.82 4.03 3.78	0.025 10 68.63 9 39.40 14.42 8.84 6.62 5.46 4.76 4.76 3.96 3.72	$\begin{array}{c} 12 \\ 76.71 & 9 \\ 39.42 \\ 14.34 \\ 8.75 \\ 6.52 \\ 5.37 \\ 4.67 \\ 4.20 \\ 3.87 \\ 3.62 \end{array}$	$\begin{array}{c} 15\\ 84.87 & 9\\ 39.43\\ 14.25\\ 8.66\\ 6.43\\ 5.27\\ 4.57\\ 4.77\\ 3.77\\ 3.52\end{array}$	20 93.10 9 39.45 14.17 8.56 6.33 5.17 4.47 4.00 3.67 3.42	24 997.25 1 39.46 14.12 8.51 6.28 5.12 4.42 3.95 3.61 3.37	30 1001.4 1 39.46 14.08 8.46 6.23 5.07 4.36 3.89 3.56 3.31	40 005.6 1 39.47 14.04 8.41 6.18 5.01 4.31 3.84 3.51 3.26	60 1009.8 1 13.99 8.36 6.12 4.96 4.25 3.78 3.45 3.20	120 014.0 1 39.49 13.95 8.31 6.07 4.90 4.20 3.73 3.39 3.14	∞ 018.3 39.50 8.26 6.02 4.85 4.14 3.67 3.33 3.08		P(F))=0.	025
F _{xspec} =3.55	f1 12 34 45 67 89 90 10 11 12 31 45	1 347.79 7 38.51 17.44 12.22 10.01 8.81 8.07 7.57 7.21 6.94 6.72 6.55 6.41 6.30 6.90	$\begin{array}{c} 2\\ 99.50 \\ 839.00\\ 10.65\\ 8.43\\ 7.26\\ 6.54\\ 6.56\\ 5.71\\ 5.46\\ 5.10\\ 4.97\\ 4.86\\ 4.76\end{array}$	$\begin{array}{c} 3\\ 64.16 \\ 839.16\\ 15.44\\ 9.98\\ 7.76\\ 6.60\\ 5.89\\ 5.42\\ 5.08\\ 4.83\\ 4.83\\ 4.63\\ 4.43\\ 4.35\\ 4.24\\ 4.15\end{array}$	4 999.58 9 39.25 15.10 9.60 7.39 6.23 5.52 5.05 4.72 4.47 4.28 4.12 4.28 4.10 3.89 3.80	5 21.85 9 39.30 9.36 7.15 5.99 4.82 4.48 4.24 4.04 3.87 3.77 3.66 3.58	6 337.11 9 39.33 14.74 9.20 6.98 5.82 5.12 4.65 5.12 4.65 4.32 4.07 3.88 3.73 3.60 3.50 3.41	7 39.362 14.62 9.07 6.85 5.70 4.99 4.53 4.20 3.95 3.95 3.61 3.48 3.348 3.29	8 56.66 9 39.37 14.54 8.98 6.76 5.60 4.43 4.10 3.85 3.66 3.51 3.39 3.20	P(F) = 9 9 39.39 39.39 14.47 8.90 6.68 5.52 4.82 4.03 3.78 3.59 3.44 3.31 3.21	0.025 10 68.63 9 39.40 14.42 8.84 6.62 5.46 4.30 3.96 3.72 3.53 3.37 3.25 3.15 3.06	$\begin{array}{c} 12\\ 76.71 & 9\\ 39.42\\ 8.75\\ 6.52\\ 5.37\\ 4.67\\ 4.20\\ 3.87\\ 3.62\\ 3.43\\ 3.28\\ 3.15\\ 3.05\\ 2.96\end{array}$	$\begin{array}{c} 15\\ 84.87 \ 9\\ 39.43\\ 14.25\\ 8.66\\ 43\\ 5.27\\ 4.57\\ 4.57\\ 4.57\\ 3.52\\ 3.38\\ 3.05\\ 2.95\\ 2.86\\ \end{array}$	20 93.10 9 39.45 6.33 5.17 4.47 4.00 3.67 3.42 3.07 2.95 2.84 2.76	24 997.25 1 39.46 6.28 5.12 4.42 3.95 3.61 3.37 3.02 2.89 2.70	30 1001.4 1 39.46 14.08 8.46 6.23 5.07 4.36 3.89 3.56 3.35 3.12 2.96 2.84 2.73 2.64	40 005.6 1 14.04 8.41 6.18 5.01 4.31 3.84 3.51 3.26 3.06 2.91 2.78 2.58	60 009.8 1 39.48 13.99 8.36 6.12 4.96 4.25 3.78 3.45 3.20 3.00 2.85 2.72 2.61 2.52	120 014.0 1 39.49 13.95 8.31 6.07 4.90 4.20 3.73 3.39 3.39 3.39 2.94 2.79 2.66 2.55 2.46	∞ 018.3 39.50 13.90 8.26 6.02 4.85 4.167 3.677 3.308 2.88 2.72 2.60 2.49 2.40		P(F))=0.	025
F _{xspec} =3.55	f ₂ f ₂ f ₃ f ₄ f ₅ f ₇ f ₇ f ₇ f ₇ f ₇ f ₇ f ₇ f ₇	1 38.51 17.44 12.22 10.01 8.81 8.07 7.57 7.21 6.72 6.55 6.41 6.30 6.20 6.20 6.20 6.04 5.982 5.92	$\frac{2}{99.500}$ 39.000 16.045 8.43 7.26 6.546 6.546 5.746 5.746 5.746 5.746 5.266 5.107 4.866 4.767 4.866 4.629 4.622 4.512	3 64.16 8 39.16 15.44 9.98 7.76 6.60 5.89 5.42 5.08 4.63 4.43 4.63 4.47 4.35 4.24 4.10 3.95 5.90 4.08 4.01 3.95 5.90 9.90	4 99.558 9 5.00 9.60 7.39 6.23 5.05 4.72 4.28 4.47 4.28 4.10 3.89 3.80 3.73 3.666 3.61 3.56 3.56	5 21.85 9 39.30 14.88 9.36 7.15 5.99 5.29 4.82 4.42 4.24 4.24 4.24 4.24 4.24 4.24	6 39.33 14.74 9.20 6.98 5.82 5.12 4.65 4.32 4.07 3.88 3.760 3.501 3.407 3.34 3.22 3.17 3.13	7 448.222 9 39.36 14.62 9.07 6.85 5.70 4.53 4.20 3.95 3.76 3.61 3.48 3.29 3.22 3.16 3.01 3.01	8 56.66 9 39.37 14.54 8.98 6.76 5.600 4.40 4.40 4.10 3.85 3.29 3.20 3.12 3.061 3.301 3.20 3.12 3.061 3.01 2.961	P(F) = 9 9 $63.28 9$ 39.39 14.49 8.90 6.68 5.52 4.82 4.30 3.78 3.59 3.44 3.31 3.21 3.05 2.98 2.84	0.025 10 68.63 9 39.40 14.42 8.84 6.62 5.46 4.76 4.30 3.96 3.72 3.53 3.57 3.15 3.15 3.15 3.16 2.99 2.87 2.87 2.87 2.87	$\begin{array}{c} 12 \\ \hline 76.71 & 9 \\ 39.42 \\ 8.75 \\ 6.52 \\ 5.37 \\ 4.67 \\ 4.67 \\ 4.20 \\ 3.87 \\ 3.62 \\ \hline 3.43 \\ 3.28 \\ 3.15 \\ 3.05 \\ 2.96 \\ 2.89 \\ 2.89 \\ 2.87 \\ 2.77 \\ 2.72 \\ 2.68 \end{array}$	$\begin{array}{c} 15\\ 84.87 & 9\\ 39.43\\ 14.25\\ 8.66\\ 6.43\\ 5.27\\ 4.57\\ 4.57\\ 4.57\\ 3.52\\ 3.38\\ 3.05\\ 2.95\\ 2.86\\ 2.79\\ 2.67\\ 2.62\\ 2.67\\ 2.62\\ 2.57\end{array}$	20 93.10 39.45 14.17 6.33 5.17 4.40 3.67 3.42 3.07 2.95 2.85 2.66 2.56 2.56 2.51	24 139.46 14.12 8.51 6.28 5.12 4.42 3.95 3.61 3.37 3.17 3.02 2.89 2.70 2.63 2.50 2.50 2.41	30 001.4 1 39.46 6.23 5.07 4.36 3.56 3.56 3.51 2.96 2.84 2.73 2.64 2.57 2.50 2.44 2.35	40 005.6 1 39.47 14.04 8.41 6.18 5.01 4.31 3.84 3.26 3.26 2.91 2.78 2.58 2.58 2.51 2.44 2.38 2.38 2.29	60 1009.8 1 39.48 13.99 8.36 6.12 4.96 4.25 3.78 3.20 3.00 2.85 2.72 2.61 2.52 2.45 2.38 2.32 2.32 2.32 2.32	$\begin{array}{c} 120\\ 014.0 \ 1\\ 39.49\\ 8.31\\ 6.07\\ 3.39\\ 3.14\\ 2.94\\ 2.79\\ 2.66\\ 2.55\\ 2.46\\ 2.38\\ 2.226\\ 2.26\\ 2.206\\ 2.16\\ \end{array}$	∞ 018.3 39.50 8.26 6.02 4.14 3.67 3.33 3.08 2.88 2.72 2.60 2.49 2.40 2.40 2.25 2.19 2.209		P(F))=0.	025
F_{xspec} =3.55 \downarrow Prob interm. between 0.05	$\begin{array}{c} f_1 \\ f_2 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 16 \\ 17 \\ 18 \\ 19 \\ 10 \\ 20 \\ 211 \\ 223 \\ 244 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ $	$\begin{array}{c} 1\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\$	$\begin{array}{c} 2\\ 999.50 \\ 839.00\\ 10.65\\ 8.43\\ 7.26\\ 6.546\\ 5.46\\ 5.46\\ 5.46\\ 5.46\\ 5.46\\ 5.46\\ 4.76\\ 4.69\\ 4.561\\ 4.46\\ 4.42\\ 4.38\\ 4.356\\ 4.32\\ 1.32\\$	3 64.16 8 39.16 1 15.44 9.98 7.76 6.60 6.60 6.60 6.592 5.42 4.43 4.43 4.47 4.13 3.95 3.86 3.86 3.88 3.85 3.87 5.777 5.77 5.77 5.77 5.77 5.77 5.77 5.7777 5.7777 5.777 5.777 5.777 5.7777 5.7777 5.7777 5.7777 5.7777 5.7777 5.7777 5.7777 5.7777 5.77777 5.7777 5.7777 5.7777 5.7777 5.77777 5.7777 5.7777 5.77777 5.77777 5.77777 5.777777 5.777777 5.7777777777	4 999.58 9 39.25 9 9.60 7.39 6.23 5.05 5.05 5.05 5.05 5.05 5.05 5.05 5.0	$\begin{array}{c} 5\\ 21.85 & 9\\ 39.30\\ 7.15\\ 5.99\\ 5.29\\ 4.48\\ 4.24\\ 4.24\\ 4.24\\ 4.24\\ 4.24\\ 3.66\\ 3.58\\ 3.$	6 339.33 9.39.33 9.20 6.98 5.82 5.82 5.82 4.63 4.07 3.860 3.50 3.41 3.341 3.24 3.22 3.13 3.05 3.05 3.05 3.299 2.997	7 48.22 \$ 39.36 9.07 6.85 5.70 4.99 4.53 4.20 3.95 3.48 3.38 3.29 3.48 3.38 3.48 3.38 3.22 3.16 3.43 3.45 3.48 3.38 3.22 3.10 3.05 3.48 3.22 3.20 4.53 3.48 3.48 3.28 5.70 4.53 3.48 3.28 5.70 5.70 4.53 3.48 3.48 3.48 3.48 3.48 3.48 3.48 3.4	8 56.66 9 39.37 14.54 4.8.98 6.76 6.76 6.76 6.76 3.31 3.39 3.29 3.20 3.30 3.29 3.29 1 2.91 2.81 2.81 2.81 2.81 2.81 2.81	P(F) = 9 63.28.9 9 14.47 8.90 6.68 5.52 4.82 4.36 4.03 3.78 3.59 3.44 3.31 3.21 3.12 3.05 2.98 2.88 2.88 2.88 2.88 2.88 2.80 2.76 2.68	0.025 10 68.63 9 39.40 6.62 5.46 4.76 6.62 3.96 3.372 3.25 3.37 3.25 3.37 2.87 2.87 2.87 2.87 2.77 2.73 2.77 2.67 2.61	$\begin{array}{c} 12\\ 76.71&9\\ 39.42\\ 8.75\\ 5.37\\ 4.67\\ 3.87\\ 3.28\\ 3.62\\ 2.89\\ 2.82\\ 2.96\\ 2.89\\ 2.82\\ 2.96\\ 2.57\\ 2.51\\ 2.5$	$\begin{array}{c} 15\\ 84.87&9\\ 39.43\\ 14.25\\ 8.66\\ 6.43\\ 5.27\\ 4.57\\ 4.57\\ 3.52\\ 3.352\\ 3.18\\ 3.05\\ 2.86\\ 2.79\\ 2.62\\ 2.57\\ 2.62\\ 2.57\\ 2.53\\ 2.50\\ 2.44\\ 2.41\\ 2.44\\ \end{array}$	20 93.10 9 339.45 8.56 6.33 5.17 4.47 3.23 3.42 2.95 2.84 2.68 2.95 2.56 2.56 2.56 2.42 2.42 2.36 2.42 2.36 2.30	24 997.25 1 3 39.46 8.51 6.28 5.12 4.425 3.61 3.37 3.17 2.89 2.89 2.89 2.89 2.89 2.20 2.41 2.30 2.41 2.31 2.30 2.24	30 001.4 1 39.46 6.23 5.07 4.36 3.56 3.56 3.51 2.73 2.57 2.50 2.57 2.57 2.57 2.57 2.57 2.24 2.31 2.27 2.21	40 005.6 1 39.47 14.8.41 6.18 5.01 4.33 3.26 3.26 2.58 2.51 2.25 2.21 2.25 2.21	60 009.8 1 33.99 8.36 6.12 4.96 6.12 4.96 5.22 2.61 2.52 2.52 2.52 2.52 2.52 2.52 2.52 2.5	$\begin{array}{c} 120\\ 014.0 \ 1\\ 39.49\\ 13.95\\ 8.31\\ 6.07\\ 4.90\\ 3.73\\ 3.39\\ 3.14\\ 2.94\\ 2.79\\ 2.76\\ 2.46\\ 2.55\\ 2.46\\ 2.38\\ 2.32\\ 2.20\\ 2.16\\ 2.38\\ 2.32\\ 2.20\\ 2.16\\ 1.98\\$	∞ 018.3 39.50 13.90 8.26 6.02 4.85 4.14 3.63 3.33 3.33 2.88 2.72 2.49 2.49 2.19 2.19 2.19 2.04 2.04 2.04 1.97 1.91		P(F))=0.	025
F _{xspec} =3.55 Prob interm. between 0.05 and 0.025	$\begin{array}{c} f_1\\ 1\\ 2\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10\\ 11\\ 12\\ 13\\ 14\\ 15\\ 6\\ 7\\ 7\\ 8\\ 9\\ 10\\ 21\\ 223\\ 24\\ 4\\ 4\\ 4\\ 25\\ 28\\ 29\\ 26\\ 7\\ 8\\ 29\\ 29\\ 26\\ 7\\ 8\\ 29\\ 9\\ 20\\ 21\\ 223\\ 24\\ 25\\ 28\\ 29\\ 29\\ 28\\ 29\\ 29\\ 28\\ 29\\ 29\\ 28\\ 29\\ 29\\ 28\\ 29\\ 29\\ 29\\ 28\\ 29\\ 29\\ 29\\ 28\\ 29\\ 29\\ 29\\ 28\\ 29\\ 29\\ 29\\ 29\\ 28\\ 29\\ 29\\ 29\\ 28\\ 29\\ 29\\ 29\\ 29\\ 29\\ 29\\ 29\\ 29\\ 29\\ 29$	$\begin{array}{c} 1\\ \\ 547.79 \\ 7.38.51\\ 10.01\\ 8.81 \\ 112.22\\ 6.94\\ 6.72 \\ 6.94\\ 6.72 \\ 6.94\\ 6.72 \\ 6.94\\ 6.25\\ 5.95\\ 5.95\\ 5.75\\$	$\begin{array}{c} 2\\ 999.50 & 8\\ 339.00\\ 10.654\\ 6.54\\ 6.54\\ 6.56\\ 6.54\\ 6.55\\ 7.11\\ 5.46\\ 4.76\\ 4.69\\ 4.62\\ 4.56\\ 4.46\\ 4.42\\ 4.32\\ 4.32\\ 4.46\\ 4.42\\ 4.22\\ 4.24\\ 4.22\\ 4.24\\ 4.22\\ 4.24\\ 4.22\\ 4.20\\ 1.24$	3 64,16,8 30,14 9,98 7,76 6,600 5,42 4,37 4,447 4,4474,447 4,447 4,447 4,4474,447 4,447 4,4474,447 4,447 4,4474,447 4,447 4,4474,447 4,447 4,4474,447 4,4474,447 4,447 4,4474,4474,447 4,447	4 99.55 9 339.25 9 9.60 9.60 9.60 9.60 9.60 4.22 4.47 4.472 4.472 4.472 4.472 4.472 4.472 4.472 4.473 3.80 3.80 3.80 3.80 3.424 3.43 3.43 3.431 3.251 9.60 9.60 9.60 9.60 9.60 9.60 9.60 9.60	$\begin{array}{c} 5\\ 21,55\\ 23,930\\ 39,30\\ 39,30\\ 39,30\\ 7,15\\ 5,29\\ 4,82\\ 4,24\\ 4,48\\ 4,44\\ 4,40\\ 4,369\\ 3,58\\ 3,37\\ 3,58\\ 3,37\\ 3,58\\ 3,37\\ 3,58\\ 3,37\\ 3,58\\ 3,37\\ 3,58\\ 3,37\\ 3,58\\ 3,37\\ 3,58\\ 3,38\\ 3,38\\ 3,38\\ 3,38\\ 3,08\\ 3,04$	6 137,119 9,20 6,58 5,82 4,65 5,12 4,02 4,02 4,02 4,02 4,02 4,02 3,37 3,34 3,24 3,24 3,24 3,29 2,99 2,997 2,987 2,988 2,987 2,988 2,998 2,997 2,998 2,998 2,998 2,988 2,998 2,997 2	7 48.22 % 9.07 6.85 5.70 9.07 6.85 5.70 6.85 5.70 8.493 4.20 3.35 3.35 3.35 3.48 3.348 3.348 3.348 3.329 3.22 3.10 3.01 3.01 2.97 2.87 2.87 2.88 2.289 2.976 2.879	8 56.66 9 9 8.98 6.76 5.60 4.40 3.85 5.60 4.40 3.85 3.66 3.51 3.39 3.20 3.12 3.01 3.39 3.20 3.12 3.01 3.29 1 2.91 2.91 2.97 2.97 2.73 2.73 2.73 2.73 2.69 2.67 2.69 2.67 2.69 2.67 2.69 2.67 2.69 2.67 2.69 2.67 2.69 2.67 2.69 2.67 2.69 2.69 2.69 2.69 2.69 2.69 2.69 2.69	$P(F) = \frac{9}{9}$ $\frac{63.28}{9.3.9} + \frac{9}{4.3.6}$ $\frac{5.52}{4.3.6} + \frac{4.36}{3.3.1}$ $\frac{3.59}{3.44} + \frac{3.3}{3.3.12}$ $\frac{3.059}{3.44} + \frac{3.3}{2.84}$ $\frac{3.059}{2.98} + \frac{2.98}{2.98}$ $\frac{2.84}{2.76}$ $\frac{2.84}{2.770}$ $\frac{2.685}{2.61} + \frac{2.67}{2.571}$	0.025 10 68.63 9 14.42 5.46 4.78 3.940 14.42 3.30 3.372 3.53 3.372 3.25 3.30 2.992 2.82 2.77 2.67 2.67 2.55 2.51 2.55	$\begin{array}{c} 12\\ 76.719\\ 8.42\\ 14.34\\ 4.57\\ 5.37\\ 4.67\\ 3.28\\ 3.42\\ 3.362\\ 3.362\\ 3.362\\ 3.362\\ 3.362\\ 2.89\\ 2.82\\ 2.96\\ 2.57\\ 2.68\\ 2.57\\ 2.64\\ 2.51\\ 2.57\\ 2.45\\ 2.57\\ 2.45\\ 2.41\\ 1.57\\ 2.45\\ 2.57\\ 2.45\\ 2.41\\ 1.57\\ 2.45\\ 2.57\\ 2.45\\ 2.45\\ 1.57\\ 2.45\\ 2.57\\ 2.45\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\ 2.45\\ 1.57\\$	$\begin{array}{c} 15\\ 84.87 & 9\\ 39.43\\ 14.25\\ 8.66\\ 6.43\\ 4.57\\ 4.57\\ 3.52\\ 2.86\\ 2.72\\ 2.67\\ 2.86\\ 2.72\\ 2.67\\ 2.47\\ 2.47\\ 2.41\\ 2.41\\ 2.41\\ 2.42\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.52\\ 2.57\\ 2.57\\ 2.57\\ 2.52\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.52\\ 2.57\\ 2.52\\ 2.57\\ $	20 93.10 9 39.45 14.17 4.00 3.67 3.42 2.84 2.84 2.84 2.84 2.84 2.84 2.84 2	24 33,462 142,251 6,28 5,12 4,42 3,95 3,61 3,95 3,37 3,37 3,37 3,37 3,37 3,37 3,37 3,3	30 001.4 1 334.66 14.46 6.23 5.07 4.36 6.23 3.59 3.59 2.24 2.26 2.39 2.30 2.26 2.39 2.31 2.27 2.41 2.27 2.41 2.27 2.41 2.27 2.21 2.27 2.21 2.07 2.	40 005.6 1 1 14.04 8.41 6.18 5.01 3.24 4.31 3.24 6.18 3.24 2.91 2.25 2.21 2.267 2.28 2.23 2.29 2.25 2.21 2.212 2.12 2.12 2.12 2.12 2.1	60 1009.9 3 3348 6.12 4.96 4.25 3.78 3.49 2.85 2.45	120 014.0 1 3.95 6.07 4.20 3.73 3.39 2.46 2.79 2.255 2.46 2.32 2.255 2.242 2.255 2.242 2.255 2.242 2.255 2.242 2.255 2.242 2.20 2.110 1.91 1.91 1.91 1.91 1.91 1.91 1.91	018.3 39.50 13.20 13.20 13.20 6.02 4.14 4.45 4.44 4.85 4.14 4.85 4.14 2.49 2.25 2.19 2.279 2.249 2.25 2.29 2.249 2.25 2.29 2.249 2.25 2.29 2.249 2.25 2.29 2.49 2.25 2.19 2.20 1.97 1.94 1.94 1.94 1.94 1.94 1.94 1.94 1.94		P(F))=0.	025
F _{xspec} =3.55 Prob interm. between 0.05 and 0.025 (actually,	$\begin{array}{c} f_1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 6 \\ 7 \\ 7 \\ 22 \\ 22 \\ 22 \\ 22 \\ 22 \\ $	$\begin{array}{c} 1\\ \\ 547.79 \\ 38.51\\ 10.01\\ 8.81 \\ 112.22\\ 6.94\\ 6.72\\ 6.94\\ 6.72\\ 6.94\\ 6.72\\ 6.94\\ 6.25\\ 5.87\\ 5.95\\ 5.75$	2 999.50 8 39.00 8.43 7.26 6.54 6.54 6.54 4.57 1.5.46 4.55 4.571 5.26 6.54 4.55 4.32 4.56 4.46 4.46 4.46 4.42 4.48 4.42 4.42 4.42 4.42 4.42 4.42	3 64,16 8 30,14 9,98 7,76 6,600 5,42 5,589 5,589 5,589 4,433 4,435 4,435 4,435 4,435 4,435 3,390 3,868 3,878	4 99.55 9 339.25 9 9.60 9.60 9.60 9.60 9.60 4.22 4.47 4.42 4.42 4.42 4.42 4.42 4.42	5 21.55 9 39.30 39.30 39.30 7.15 5.99 4.82 4.24 4.48 4.48 4.44 4.40 4.40 4.30 3.57 3.58 3.37 3.58 3.33 3.29 3.25 3.13 3.10 3.00 3.25 3.25 2.3.18 3.10 3.25 3.25 3.25 3.25 3.25 3.25 3.25 3.25	6 137,119 14,74 9,20 6,582 5,82 4,65 5,12 4,05 4,05 4,05 3,07 3,07 3,07 3,07 3,07 3,07 3,07 3,07 3,07 3,07 3,07 4,02 4,05	7 48,22 % 39,36 14,62 4,53 3,95 6,85 5,70 9,07 6,85 5,70 4,53 3,95 3,01 3,22 3,10 3,48 3,348 3,348 3,348 3,329 3,22 3,10 3,01 3,01 2,97 2,87 2,87 2,87 2,87 2,87 2,87 2,87 2,8	8 56.66 9 9 8.98 6.76 5.80 4.40 3.85 5.60 4.40 3.85 3.39 3.20 3.12 3.39 3.20 3.12 2.91 2.91 2.91 2.91 2.91 2.97 2.91 2.97 2.73 2.73 2.73 2.73 2.45 2.53 2.41 2.30	$\begin{array}{c} P(F) = \\ 9 \\ \hline \\ 63.28 & 9 \\ 39.39 \\ 14.47 \\ 8.90 \\ 6.68 \\ 8.90 \\ 6.68 \\ 3.31 \\ 3.59 \\ 3.41 \\ 3.31 \\ 3.12 \\ 3.31 \\ 3.21 \\ 3.31 \\ 3.21 \\ 3.42 \\ 2.84 \\$	0.025 10 68.63 9 39.40 14.42 5.46 4.73 3.53 3.37 3.53 3.37 3.53 3.37 2.82 2.77 2.67 2.67 2.55 2.53 2.53 2.39 2.23 2.21	$\begin{array}{c} 12\\ 76.719\\ 8.42\\ 14.34\\ 4.67\\ 3.42\\ 3.42\\ 3.43\\ 3.28\\ 2.96\\ 2.53\\ 2.96\\ 2.57\\ 2.96\\ 2.57\\ 2.66\\ 2.57\\ 2.45$	15 884.87 9 14.25 8.669 6.43 14.25 5.27 4.57 4.57 3.33 3.18 2.869 2.79 2.467 2.2867 2.2867 2.4777 2.4777 2.4777 2.47777 2.47777777777	$\begin{array}{c} 20\\ 93.10 & \\ 33.45 \\ 33.45 \\ 33.45 \\ 4.7 \\ 3.307 \\ 2.84 \\ 3.23 \\ 3.07 \\ 2.84 \\ 2.84 \\ 2.84 \\ 2.83 \\ 2.83 \\ 2.33 \\ 2.33 \\ 2.33 \\ 2.33 \\ 2.24 \\ 2.42 \\ 2.26 \\ 2.42 \\ 2.23 \\ 2.23 \\ 2.21 \\ 2.42 \\ 2.20 \\ 1.92 \\ 2.20 \\ 1.92 \\ 1.82 \end{array}$	24 33,461 33,462 14,52 6,28 5,12 4,42 4,42 3,95 3,61 3,95 3,37 3,37 3,37 3,37 3,395 3,37 3,395 3,37 3,395 3,37 3,395 2,50 2,50 2,50 2,50 2,50 2,50 2,50 2,5	30 001.4 1 39.46 6.23 5.07 4.36 6.23 3.59 3.22 2.94 2.39 2.30 2.34 2.50 2.34 2.37 2.44 2.57 2.39 2.34 2.37 2.44 2.57 2.39 2.31 2.11 2.07 1.42 2.64 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.11 2.07 1.41 2.11 2.11 2.11 2.07 1.41 2.11 2.11 2.11 2.11 2.07 1.41 2.11 2.11 2.11 2.11 2.07 1.41 2.11 2.11 2.07 1.41 2.11 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.11 2.07 1.41 2.07 1.41 2.11 2.07 1.42 2.07 1.41 2.07 1.42 2.07 1.42 2.07 1.42 2.07 1.44 1.45 2.07 1.45	40 005.6 1 133.47 14.4 6.18 6.18 3.51 3.64 3.06 2.91 2.25 2.58 2.51 2.44 2.38 2.25 2.21 2.44 2.33 2.29 2.25 2.21 2.22 2.21 2.25 2.15 2.15 2.20 2.01 1.88 1.74 1.14 1.14 2.15 2.15 2.15 2.15 2.15 2.15 2.15 2.15	60 1000.8 3 33948 33948 6.12 4.966 4.255 3.768 3.788 3.200 2.855 2.455 2.452 2.4	$\begin{array}{c} 120\\ 014.0 \\ 13.95\\ 3.95\\ 3.39\\ 4.90\\ 4.20\\ 3.73\\ 3.39\\ 2.66\\ 2.32\\ 2.25\\ 2.46\\ 2.32\\ 2.26\\ 2.16\\ 1.91\\ 1.91\\ 1.91\\ 1.91\\ 1.87\\ 1.75\\ 1.58\\ 1.43\\ \end{array}$	018.3 39.50 13.90 13.26 6.02 4.14 4.85 4.14 4.85 4.14 2.85 2.60 2.49 2.279 2.249 2.25 2.19 2.249 2.25 2.29 2.249 2.25 2.19 2.29 2.249 2.25 2.19 2.20 1.97 1.94 1.88 1.81 1.81 1.79		P(F))=0.	025
F _{xspec} =3.55 Prob interm. between 0.05 and 0.025 (actually, 0.0323)	$ \begin{array}{c} f_1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \\ 15 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 112 \\ 12 \\ 22 \\ 23 \\ 24 \\ 25 \\ 26 \\ 6 \\ 29 \\ 30 \\ 0 \\ 120 \\ 0 \\ 120 \\ 0 \\ 8 \\ 5 \\ 6 \\ 7 \\ 7 \\ 28 \\ 29 \\ 30 \\ 0 \\ 120 \\ 0 \\ 8 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5 \\ 5$	1 347.79 7 38.51 10.01 8.81 18 8.81 18 8.81 18 6.52 6.41 6.54 6.54 6.54 6.54 6.54 6.54 6.54 7.54 7.55 7.55 7.55 7.55 7.55 7.55 7	2 999.50 8 33.00 6.54 6.54 6.54 6.54 6.54 6.54 6.54 6.54	3 64.16 8 39.14 19.98 7.76 6.60 5.42 5.42 4.63 4.63 4.43 4.43 4.43 4.43 4.43 4.45 4.08 4.35 3.396 3.396 3.386 3.87 3.57 3.67 3.61 3.567 3.521 3.567 3.522 3.67 3.523 3.521 3.525 3.	4 99,559 339,259 9,600 7,399 6,233 5,552 5,552 5,552 4,4744,472 4	5 21.55 9.36 14.83 9.36 7.15 5.99 5.22 4.82 4.484 4.24 3.377 3.66 3.53 3.53 3.53 3.53 3.52 3.18 3.10 3.06 3.06 3.04 3.013 3.00 3.02 2.57 able. 2.57	$\begin{array}{c} 6\\ \hline \\ 37,119\\ 9,33\\ 14,74\\ 4,55\\ 8,20\\ 6,98\\ 4,452\\ 4,07\\ 4,32\\ 4,07\\ 4,32\\ 4,07\\ 3,36\\ 3,50\\ 3,50\\ 3,41\\ 3,34\\ 3,28\\ 3,60\\ 3,50\\ 3,41\\ 3,34\\ 3,28\\ 3,60\\ 3,50\\ 3,41\\ 3,34\\ 3,28\\ 2,99\\ 2,97\\ 2,94\\ 2,92\\ 2,90\\ 2,90\\ 2,97\\ 2,94\\ 2,92\\ 2,90\\ 2,90\\ 2,92\\ 2,90\\ 2,92\\ 2,90\\ 2,92\\ 2,90\\ 2,92\\ 2,90\\ 2,92\\ 2,90\\ 2,92\\ $	7 48.22 \$ 9.07 6.85 5.70 9.07 6.85 5.70 3.95 3.95 3.48 3.348 3.348 3.329 3.22 3.10 0 3.05 3.05 3.05 2.97 2.87 2.87 2.87 2.82 2.88 2.78 2.78 2.49 3.22 2.29	8 56.66 9 8.98 6.76 5.60 3.51 3.66 3.51 3.39 3.29 3.29 3.20 3.12 3.30 3.20 3.12 3.30 3.20 3.12 3.30 3.20 3.12 3.30 3.20 3.12 3.30 3.20 3.12 3.00 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.12 5.60 3.01 3.29 3.20 3.20 3.20 3.20 3.20 3.20 3.20 3.20	$\begin{array}{c} P(F) = \\ 9 \\ \hline \\ 8 \\ 30.39 \\ 14.47 \\ 8.90 \\ 6.68 \\ 8.90 \\ 6.68 \\ 3.39 \\ 14.87 \\ 3.59 \\ 3.41 \\ 3.51 \\ 3.31 \\ 3.11 \\ 3.12 \\ 3.33 \\ 3.12 \\ 2.98 \\ 2.88 \\$	0.025 10 68.63 9 14.42 8.84 4.76 5.46 4.73 3.37 3.37 3.37 3.37 3.37 3.37 2.53 3.06 2.99 2.97 2.64 2.64 2.55 2.55 2.51 2.39 2.57 2.55 2.51 2.59 2.57 2.55 2.51 2.59 2.55 2.51 2.55 2.51 2.55 2.5	$\begin{array}{c} 12\\ 76.71 & 9\\ 3.42\\ 14.34\\ 8.75\\ 6.52\\ 3.42\\ 3.36\\ 3.28\\ 2.36\\ 3.36\\ 3.38\\ 2.36\\ 2.39\\ 2.96\\ 2.57\\ 2.96\\ 2.57\\ 2.54\\ 2.57\\ 2.45\\ 2.54\\ 2.51\\ 2.54\\ 2.55\\ 2.49\\ 2.47\\ 2.55\\ 1.94\\ 1$	$\begin{array}{c} 15\\ 84.87 & 9\\ 89.43\\ 14.25\\ 84.66\\ 43.33\\ 3.14\\ 2.57\\ 4.57\\ 3.52\\ 2.57\\ 2.86\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.57\\ 2.47\\ 2.57\\ 2.59\\ 2.52\\ 2.57\\ 2.47\\ 2.41\\ 2.18\\ 2.39\\ 2.34\\ 2.31\\ 2.18\\ 2.06\\ 1.94\\ 1.83\\ 3.52$	20 93.10 § 33.45 33.45 6.33 5.17 4.47 4.47 3.42 2.55 2.46 2.25 2.25 2.25 2.42 2.25 2.42 2.29 2.33 2.30 2.33 2.30 2.33 2.30 2.42 2.22 2.23 2.20 1.22 2.22 2.23 2.21 1.22 2.22 2.23 2.23 2.21 2.22 2.23 2.23 2.23 2.23 2.23 2.23 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.25 2.23 2.24 2.24 2.25 2.23 2.24 2.24 2.25 1.24 2.24 2.25 1.24 2.25 1.24 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.25 1.24 2.42 2.24 2.25 1.24 2.42 2.24 2.25 1.24 2.24 2.25 1.24 2.24 2.25 1.24 2.25 1.24 2.25 1.24 2.25 1.24 2.25 1.24 2.25 1.24 2.25 1.24 2.25 2.23 2.21 2.20 2.27 2.27 2.20 2.27 2.27 2.20 2.27 2.20 2.27 2.27 2.27 2.27 2.27 2.20 2.27 2.2	24 197.25 1 339.45 14.12 1	30 001.4 1 39.46 14.08 14.08 6.23 5.07 4.36 6.23 3.89 3.33 2.64 2.50 2.44 2.50 2.44 2.50 2.44 2.51 2.50 2.44 2.27 2.50 2.44 2.21 2.50 2.44 2.51 2.50 1.25 1.	40 005.6 1 33.47 14.04 6.18 5.01 4.31 3.84 3.326 2.53 2.291 2.44 2.33 2.291 2.44 2.33 2.291 2.44 2.33 2.291 2.44 2.33 2.291 2.44 2.33 2.291 2.44 2.33 2.291 2.44 1.15 2.15 2.15 2.15 2.15 2.15 2.15 1.15 2.15 1.15 1.16 1.17 1.16 1.17 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.74 1.75	60 139.48 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 13.99 2.95 2.61 2.52 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.32 2.45 2.38 2.39 2.00 3.0	$\begin{array}{c} 120\\ 014.0 \\ 1\\ 39.49\\ 13.95\\ 13.95\\ 13.95\\ 13.95\\ 2.85\\ 2.85\\ 2.85\\ 2.92\\ 2.10\\ 2.55\\ 2.46\\ 2.32\\ 2.26\\ 2.25\\ 2.46\\ 2.32\\ 2.26\\ 2.10\\ 1.21\\ 1.21\\ 1.12\\ 1.$	∞ 018.3 39.50 8.26 6.02 4.14 3.33 3.67 3.33 2.86 2.82 2.49 2.42 2.25 2.19 2.42 2.04 2.04 2.04 2.04 2.04 2.04 2.04 1.97 1.94 1.81 1.41 1.64 1.31 1.00	101	P(F))=0.	025

=0.025

Errors within XSPEC: the contour plots



Confidence	sigma	delta_chi-square
68.3%	1.0	1.00
90.0%	1.6	2.71
95.5%	2.0	4.00
99.0%	2.6	6.63
99.7%	3.0	9.00

- Ex.1: Error at 90% confidence level for one parameter of interest: xspec> error #param 2.71
- Ex. 2: Error at 90% confidence level for two parameters of interest: xspec> error #param 4.61
- Ex. 3: Error at 99% confidence level for one parameter of interest: xspec> error #param 6.63



	- ΤΑ χ ² (BLE F distributi	on critic	al value	25								
	Tail probability <i>p</i>												
	df	.25	.20	.15	.10	.05	.025	.02	.01	.005	.0025	.001	.0005
Parameter of interest ——	► 1 2 3 4 5	1.32 2.77 4.11 5.39 6.63	1.64 3.22 4.64 5.99 7.29	2.07 3.79 5.32 6.74 8.12	2.71 4.61 6.25 7.78 9.24	3.84 5.99 7.81 9.49 11.07	5.02 7.38 9.35 11.14 12.83	5.41 7.82 9.84 11.67 13.39	6.63 9.21 11.34 13.28 15.09	7.88 10.60 12.84 14.86 16.75	9.14 11.98 14.32 16.42 18.39	10.83 13.82 16.27 18.47 20.51	12.12 15.20 17.73 20.00 22.11

Table entry for *p* is the critical value $(\chi^2)^*$ with probability *p* lying to its

right.

Low-counting statistic regime

The fit statistic routinely used is referred to as the χ^2 statistic

$$S = \sum_{i} \left(\frac{S_{i} - B_{i}t_{s}}{O_{k}} / t_{b} - \frac{m_{i}t_{s}}{E_{k}} \right)^{2} / \left(\left(\frac{\sigma_{s}}{\sigma_{s}} \right)_{i}^{2} + \left(\frac{\sigma_{B}}{\sigma_{k}} \right)_{i}^{2} \right)$$

where $S_i = \text{src counts in the } I=\{1,...,N\}$ data bins with exposure t_S , $B_i = \text{background counts with exposure } t_B$ and $m_i = \text{model predicted count rate;}$ $(\sigma_S)^2$ and $(\sigma_B)^2 = \text{variance on the src and background counts, typically estimated by <math>S_i$ and B_i

BUT

the χ^2 statistic fails in low-counting regime (few counts in each data bin) Possible solution: to rebin the data so that each bin contains a large enough number of counts BUT

 \Rightarrow Loss of information and dependence upon the <u>rebinning method</u> adopted

Viable solution: to modify S so that it performs better in low-count regime

⇒ Estimate the variance for a given data bin by the average counts from surrounding bins (Churazov et al. 1996)

BUT

need for Monte-Carlo simulations to support the result

The Cash statistic

Construct a maximum-likelihood estimator based on the Poisson distribution of the detected counts

(Cash 1979; Wachter et al. 1979) in presence of background counts \Rightarrow implemented into XSPEC

Finding the maximum likelihood means finding the best fit of parameters that maximize the Poisson likelihood



Spectrum fitted using the **C-stat** and then rebinned just for presentation purposes

• χ^2 statistics \Leftrightarrow Gaussian statistics

C-statistics Poisson statistics

How does statistics enter into X-ray spectral fitting?

(adapted from K. Arnaud presentation; see also "Handbook of X-ray Astronomy", edited by K. Arnaud, R. Smith, A. Siemiginowska)

Forward-fitting

The standard method of analyzing X-ray spectra is "forward-fitting". This comprises the following steps...

- Calculate a model spectrum.
- Multiply the result by an instrumental response matrix.
- Compare the result with the actual observed data by calculating some statistic.
- Modify the model spectrum and repeat till the best value of the statistic is obtained.



This only works if the model spectrum can be expressed in a reasonably small number of parameters (although I have seen people fit spectra using models with over 100 parameters).

The aim of the forward-fitting is then to obtain the best-fit and confidence ranges of these parameters.

Spectral fitting programs

o **XSPEC** - part of HEAsoft. General spectral fitting program with many models available.

o Sherpa - part of CIAO. Multi-dimensional fitting program which includes the XSPEC model library and can be used for spectral fitting.

o SPEX - from SRON in the Netherlands. Spectral fitting program specialising in collisional plasmas and high resolution spectroscopy.

o ISIS - from the MIT Chandra HETG group. Mainly intended for the analysis of grating data. Incorporated in Sherpa as GUIDE.

Models

All models are wrong, but some are useful - George Box

X-ray spectroscopic models are usually built up from individual components. These can be thought of as two basic types -additive (an emission component e.g. blackbody, line,...) or multiplicative (something which modifies the spectrum e.g. absorption).

Model = $M_1 * M_2 * (A_1 + A_2 + M_3 * A_3) + A_4$

Additive Models

Basic additive (emission) models include :

- blackbody
- thermal bremsstrahlung
- power-law
- collisional plasma
- Gaussian or Lorentzian lines

There are many more models available covering specialised topics such as accretion disks, comptonized plasmas, non-equilibrium ionization plasmas, multitemperature collisional plasmas...

Multiplicative (and other) Models

and multiplicative models include :

- photoelectric absorption due to our Galaxy
- photoelectric absorption due to ionized material
- high energy exponential roll-off.
- edge with $1/E^3$ roll-off.

XSPEC also has a couple of other types of model components (convolution, mixing) which are used like a multiplicative model but perform more complicated operations on the current model.

Finding the best-fit (I)

Finding the best-fit means minimizing the statistic value. There are many algorithms available to do this in a computationally efficient fashion (see Numerical Recipes).

Most methods used to find the best-fit are local i.e. they use some information around the current parameters to guess a new set of parameters. All these methods are liable to get stuck in a local minimum. Watch out for this !

The more complicated your model and the more highly correlated the parameters then the more likely that the algorithm will not find the absolute best-fit.

Finding the best-fit (II)

Sometimes you can spot that you are stuck in a local minimum by using the XSPEC **error** or **steppar** commands. These both step through parameter values, error in the vicinity of the current best-fit and steppar over a user-defined grid, and thus can stumble across a better fit. Crude but sometimes effective.

You can do this in a semi-automated fashion by using a local minimization algorithm and following this with the error command with the ability to restart if a new minimum is found during the search.



The more complicated the model and the more highly correlated the parameters, then the more likely that the algorithm will hardly find the true minimum

Dealing with background

o Unless you are looking at a bright point source with *Chandra*, you will probably have a background component to the spectrum in addition to the source in which you are interested.

• You can include background in the model but this is complicated and is not usually used.

o The usual method is to extract a spectrum from another part of the image or another observation. Spectral fitting programs then use both the source and background spectra.

o If the background spectrum is extracted from a different sized region than the source, then the background spectrum is scaled by the spectral fitting program (using the BACKSCAL keyword in the FITS file).

Spectra with few counts

o Be careful if you have few photons/bin. χ^2 is biased in this case with fluctuations below the model having more weight than those above, causing the fit model to lie below the true model.

o A common solution is to bin up your spectrum so all the bins have > some number of photons. Don't do this - it loses information and introduces a bias that is difficult to quantify.

o Solutions are to use a different weighting scheme (e.g., *weight churazov* option in XSPEC) or a maximum likelihood statistic (the "C-statistic" - *stat cstat* in XSPEC).

o The problem with these options is that while they give best fit parameters they do not provide a goodness-of-fit measure.

Final advices and admonitions

- Remember that the purpose of spectral fitting is to attain understanding, not fill up tables of numbers
- Physics behind X-ray spectral modeling is fundamental; then statistics needed to support the results
- Don't misuse the F-test, and use confidence contours
- Try to test whether you really have found the best-fit

and always remember ...

"There are three sorts of lies: lies, damned lies and statistics"