A taste of statistics and applications to X-ray spectral fitting

- ✓ Normal error (Gaussian) distribution

  → most important in statistical analysis of data, describes the distribution of random observations for many experiments

- ✓ Poisson distribution

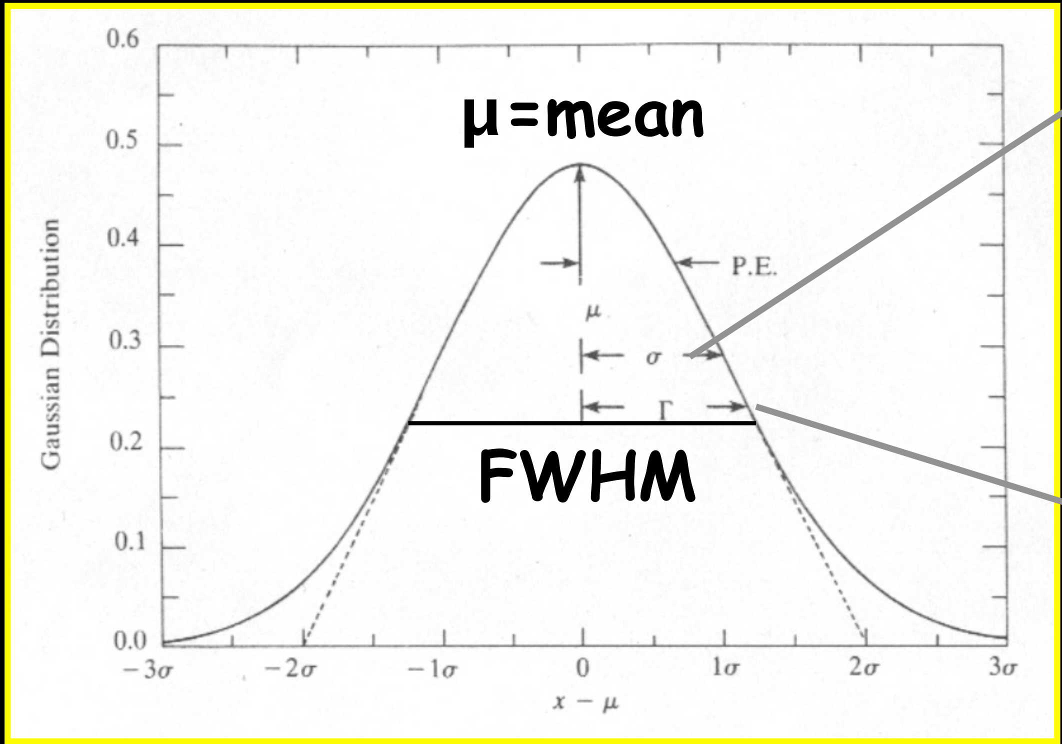
  → generally appropriate for counting experiments related to random processes (e.g., radioactive decay of elementary particles)

- ✓ Statistical tests: χ^2 and F-test

Further details in the XSPEC presentation

The Gaussian (normal error) distribution

Casual errors are above and below the "true" (most "common") value
→ bell-shape distribution if systematic errors are negligible



σ=standard deviation of the function

Γ=half-width at half maximum=1.17σ

FWHM=2.35σ

Gaussian probability function

$$P(x;\ \mu,\ \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Probability Density Function
(centered on μ)

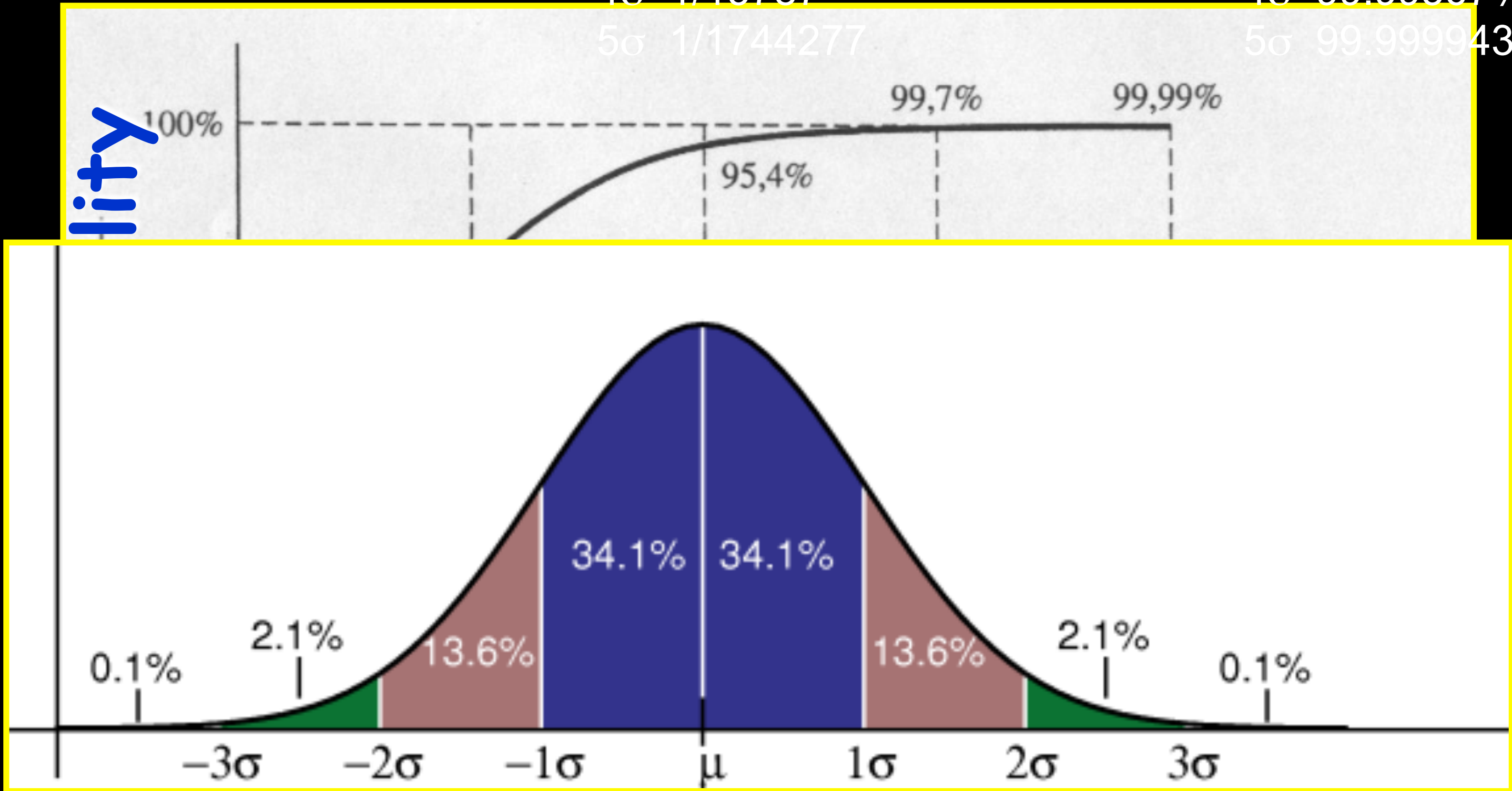
μ=mean value
σ=standard deviation

normalization factor, so that ∫f(x) dx=1

$$e^{-x^2/2\sigma^2}$$

function centered on 0

1σ	1/3	1σ 68.3%
2σ	1/22	2σ 95.45%
3σ	1/370	3σ 99.730%
4σ	1/15787	4σ 99.99367%
5σ	1/1744277	5σ 99.999943%



σ	0	0,25	0,5	0,75	1,0	1,25	1,5	1,75	2,0	2,5	3,0	3,5	4,0
P (%)	0	20	38	55	68	79	87	92	95,4	98,8	99,7	99,95	99,99

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} dx$$

Cumulative Distribution Function

The Poisson distribution

Describes experimental results where events are counted and the uncertainty is not related to the measurement but reflects the intrinsically casual behavior of the process (e.g., radioactive decay of particles, X-ray photons, etc.)

$$\mathrm{P}(x) = e^{-\mu}\mu^{x}/x! \quad (x=0,1,2,\ldots)$$

Probability of obtaining x events when the expected number is μ

μ>0: main parameter of the Poisson distribution

x=observed number of events in a time interval (frequency of events)

average number of events

$$\bar{x} = \sum_{x=0}^{\infty} xP(x) = \sum_{x=0}^{\infty} xe^{-\mu}\mu^{x}/x! = \mu$$

➔μ=average number of expected events if the experiment is repeated many times

$$\sigma^2 = \langle (x-\mu)^2 \rangle =$$

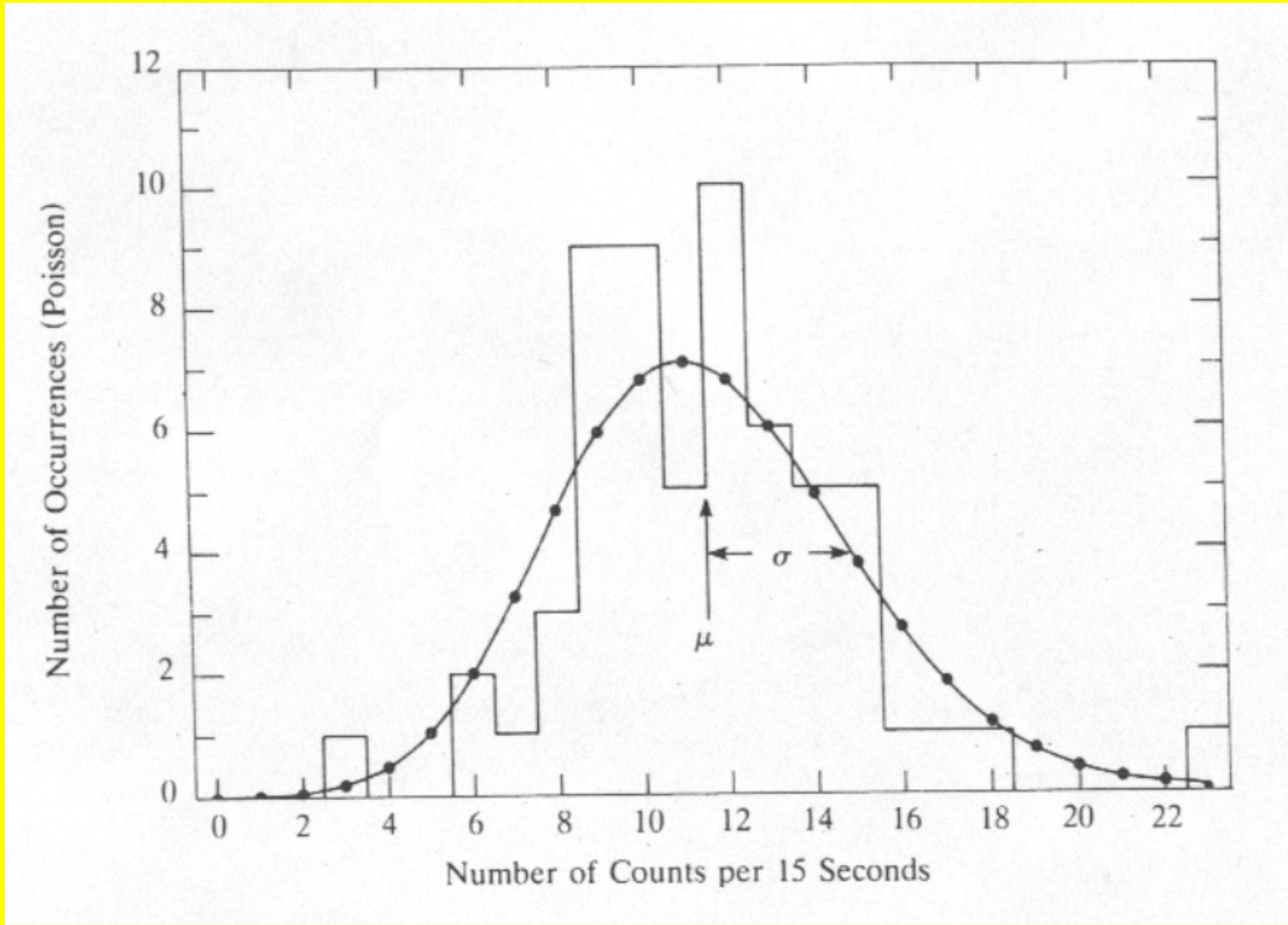
expectation value of the square of the deviations

$$= \sum_{x=0}^{\infty} (x-\mu)^2 \frac{\mu^x}{x!} e^{-\mu} = \mu$$

the Poisson distribution with average counts=μ has standard deviation √μ

μ » : the Poisson distribution is approximated by the Gaussian distribution

defined by only one parameter μ



F-test

If two statistics following the χ^2 distribution have been determined, the ratio of the reduced chi-squares is distributed according to the F distribution

$$P_f(f;\upsilon_1,\upsilon_2) = \frac{\chi_1^2/\upsilon_1}{\chi_2^2/\upsilon_2}$$

$$\propto \Delta\chi^2/k$$

with k=number of additional terms (parameters)

Example: Use the F-test to evaluate the improvement to a spectral fit due to the assumption of a different model, with additional terms

Conditions: (a) the simpler model is nested within the more complex model; (b) the extra parameters have Gaussian distribution (not truncated by the parameter space boundaries)

→see the F-test tables for the corresponding probabilities (specific command in XSPEC)

An application of the F-test

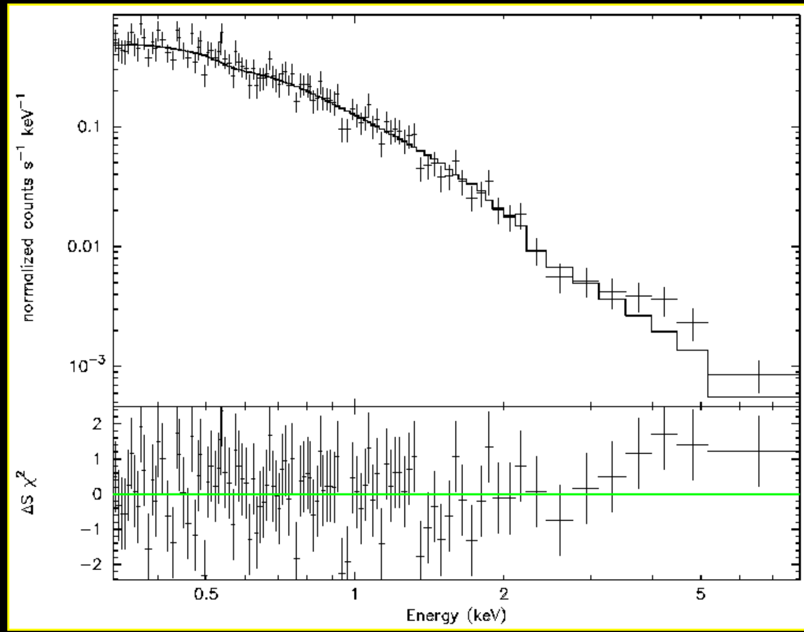
```
========================================================================
Model phabs<1>*powerlaw<2> Source No.: 1   Active/On
Model Model Component  Parameter  Unit     Value
 par  comp
   1    1   phabs      nH         10^22    1.59000E-02  frozen
   2    2   powerlaw   PhoIndex            2.72811      +/-  0.0
   3    2   powerlaw   norm                1.51490E-04  +/-  0.0
________________________________________________________________________

  Using energies from responses.

 Chi-Squared =            97.23 using 105 PHA bins.
 Reduced chi-squared =          0.9440 for    103 degrees of freedom
 Null hypothesis probability =    6.417127e-01
```

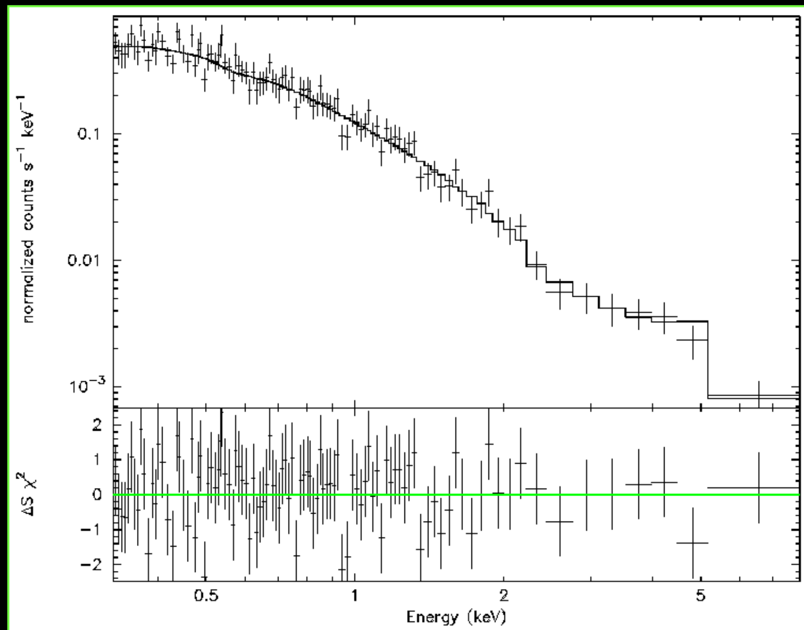
Model1

```
========================================================================
Model phabs<1>(laor<2> + powerlaw<3>) Source No.: 1   Active/On
Model Model Component  Parameter  Unit     Value
 par  comp
   1    1   phabs      nH         10^22    1.59000E-02  frozen
   2    2   laor       lineE      keV      5.23582      +/-  0.0
   3    2   laor       Index               3.00000      frozen
   4    2   laor       Rin(G)              1.23500      frozen
   5    2   laor       Rout(G)             400.000      frozen
   6    2   laor       Incl       deg      30.0000      frozen
   7    2   laor       norm                6.83065E-06  +/-  0.0
   8    3   powerlaw   PhoIndex            2.77137      +/-  0.0
   9    3   powerlaw   norm                1.48123E-04  +/-  0.0
________________________________________________________________________

  Using energies from responses.

 Chi-Squared =            90.84 using 105 PHA bins.
 Reduced chi-squared =          0.8994 for    101 degrees of freedom
 Null hypothesis probability =    7.557789e-01
 Current data and model not fit yet.
 Weighting method: standard
XSPEC12>ftest 90.84 101 97.2 103
 F statistic value = 3.53567 and probability 0.0327981
```

Model1+extra component

low F value ⇒ likely low signif. of add. comp





$$F_t = (\frac{\chi^2(dof) - \chi^2(dof-k)}{dof-(dof-k)})/(\chi^2(dof-k)/(dof-k)) =$$

$$=(\Delta\chi^2/k)/\chi^2_\nu$$

Ex: $\chi^2(103) = 97.23$

$\chi^2(101) = 90.84$

$\rightarrow \Delta\chi^2 = 6.39, k = 2 \rightarrow F_t = (6.39/2)/(90.84/101) = 3.55$

F_t follows the F distribution with v_1=k=Δ(dof) and v_2=dof-k(-1)

➔ Search in the F-distribution tables for the probability of the null hypothesis (H_0) for v_1=2 and v_2=100

$v_1=2$
$v_2=100$
(60−120)

F=3.15,3.07
at P(F)=0.05

F=3.93,3.80
at P(F)=0.025

$F_{xspec}=3.55$

⇩

Prob interm. between 0.05 and 0.025 (actually, 0.0323)

P(F)=0.05

TABLE 5 (Contd.)

$P(F) = 0.05$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54	241.88	243.91	245.95	248.01	249.05	250.09	251.14	252.20	253.25	254.32
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.84	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.00	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	[illegible]	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

P(F)=0.025

$P(F) = 0.025$

f_2 \ f_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	647.79	799.50	864.16	899.58	921.85	937.11	948.22	956.66	963.28	968.63	976.71	984.87	993.10	997.25	1001.4	1005.6	1009.8	1014.0	1018.3
2	38.51	39.00	39.16	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.42	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
3	17.44	16.04	15.44	15.10	14.88	14.74	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25	4.20	4.14
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49
15	6.20	4.76	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.58	2.52	2.46	2.40
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95	1.88
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00	1.93	1.85
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91	1.83
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89	1.81
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
60	[illegible]	3.93	[illegible]	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
120	[illegible]	3.80	[illegible]	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00

See notes at the beginning of the table.