BEAMING, SYNCHROTRON AND INVERSE COMPTON

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Chapter 1

Beaming

1.1 Rulers and clocks

Special relativity taught us two basic notions: comparing dimensions and flow of times in two different reference frame, we find out that they differ. If we measure a ruler at rest, and then measure the same ruler when is moving, we find that, when moving, the ruler is shorter. If we syncronize two clocks at rest, and then let one move, we see that the moving clock is delaying. Let us see how this can be derived by using the Lorentz transformations, connecting the two reference frames K (that sees the ruler and the clock moving) and K' (that sees the ruler and the clock at rest). For semplicity, but without loss of generality, consider a a motion along the x axis, with velocity $v \equiv \beta c$ corresponding to the Lorentz factor Γ. Primed quantities are measured in K'. We have:

$$
\begin{aligned}
x' &= \Gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \Gamma\left(1 - \beta\frac{x}{c}\right)
\end{aligned}
\tag{1.1}
$$

with the inverse relations given by

$$
\begin{aligned}
x &= \Gamma(x' + vt') \\
y &= y' \\
z &= z' \\
t &= \Gamma\left(t' + \beta\frac{x'}{c}\right).
\end{aligned}
\tag{1.2}
$$

The length of a moving ruler has to be measured through the position of its extremes *at the same time t*. Therefore, as $\Delta t = 0$, we have

$$
x_2' - x_1' = \Gamma(x_2 - x_1) - \Gamma v \Delta t = \Gamma(x_2 - x_1)
\tag{1.3}
$$

i.e.

$$\Delta x = \frac{\Delta x'}{\Gamma} \rightarrow \text{contraction} \qquad (1.4)$$

Similarly, in order to determine a time interval a (lab) clock has to be compared with one in the comoving frame, which has, in this frame, *the same position x'*. Then

$$\Delta t = \Gamma \Delta t' + \Gamma \beta \Delta \frac{x'}{c} = \Gamma \Delta t' \rightarrow \text{dilation} \qquad (1.5)$$

An easy way to remember the transformations is to think to mesons produced in collisions of cosmic rays in the high atmosphere, which can be detected even is their lifetime (in the comoving frame) is much shorter than the time needed to reach the earth's surface. For us, on ground, relativistic mesons live longer (for the meson's point of view, instead, it is the length of the travelled disctance which is shorter).

All this is correct if we measure lengths by comparing rulers (at the same time in K) and by comparing clocks (at rest in K') – the meson lifetime *is* a clock. In other words, **if we do not use photons** for the measurement process.

1.2   Photographs and light curves

If we have an extended moving object and if the information (about position and time) are carried by photons, we **must** take into account their (different) travel paths. When we take a picture, we detect photons arriving at the same time to our camera: if the moving body which emitted them is extended, we must consider that these photons have been emitted at different times, when the moving object occupied different locations in space. This may seem quite obvious. And it is. Nevertheless these facts were pointed out in 1959 (Terrel 1959; Penrose 1959), more than 50 years after the publication of the theory of special relativity.

1.2.1   The moving bar

Let us consider a moving bar, of proper dimension ℓ', moving in the direction of its length at velocity βc and at an angle θ with respect to the line of sight (see Fig. 1.1). The length of the bar in the frame K (according to relativity "without photons") is $\ell = \ell'/\Gamma$. The photon emitted in A_1 reaches the point H in the time interval Δt_e. After Δt_e the extreme B_1 has reached the position B_2, and by this time, photons emitted by the other extreme of the bar can reach the observer simultaneously with the photons emitted by A_1, since the travel paths are equal. The length $B_1 B_2 = \beta c \Delta t_e$, while $A_1 H = c \Delta t_e$. Therefore

$$A_1 H = A_1 B_2 \cos\theta \rightarrow \Delta t_e = \frac{\ell' \cos\theta}{\Gamma(1 - \beta\cos\theta)}. \qquad (1.6)$$
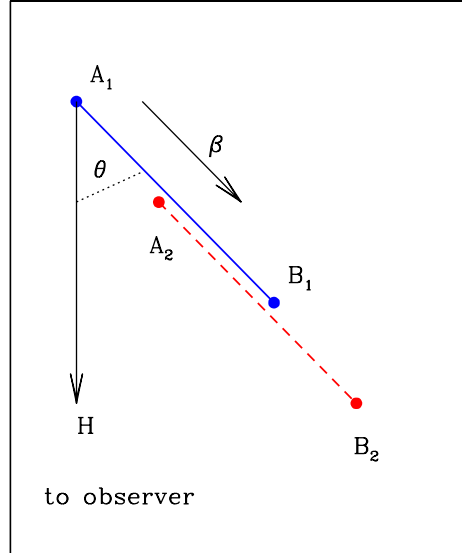
Figure 1.1: A bar moving with velocity βc in the direction of its length. The path of the photons emitted by the extreme A is longer than the path of photons emitted by B. When we make a picture of the bar (or a map), we collect photons reaching the detector simultaneously. Therefore the photons from A have to be emitted before those from B, when the bar occupied another position.

Note the appearance of the term $\delta = 1/[\Gamma(1-\beta\cos\theta)]$ in the transformation: this accounts for both the relativistic length contraction $(1/\Gamma)$, and the Doppler effect $[1/(1-\beta\cos\theta)]$. The length A_1B_2 is then given by

$$A_1B_2 = \frac{A_1H}{\cos\theta} = \frac{\ell'}{\Gamma(1-\beta\cos\theta)} = \delta\ell'. \tag{1.7}$$

In a real picture, we would see the projection of A_1B_2, i.e.:

$$HB_2 = A_1B_2\sin\theta = \ell'\frac{\sin\theta}{\Gamma(1-\beta\cos\theta)} = \ell'\delta\sin\theta, \tag{1.8}$$

The observed length depends on the viewing angle, and reaches the maximum (equal to ℓ') for $\cos\theta = \beta$.

1.2.2 The moving square

Now consider a square of size ℓ' in the comoving frame, moving at 90° to the line of sight (Fig. 1.2). Photons emitted in A, B, C and D have to arrive
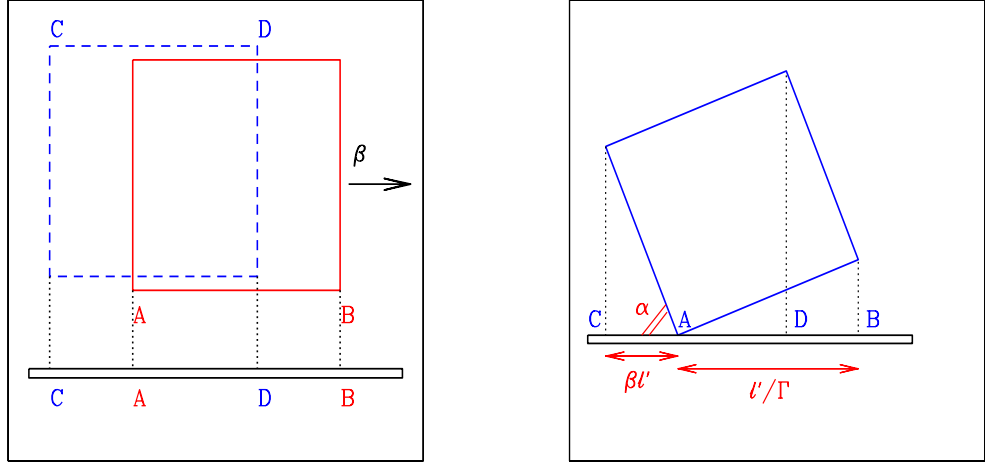
Figure 1.2: *Left:* A square moving with velocity βc seen at $90°$. The observer can see the left side (segment CA). Light rays are assumed to be parallel, i.e. the square is assumed to be at large distance from the observer. *Right:* The moving square is seen as *rotated* by an angle α given by $\cos\alpha = \beta$.

to the film plate at the same time. But the paths of photons from C and D are longer $\rightarrow$ they have to be emitted earlier than photons from A and B: when photons from C and D were emitted, the square was in another position.

The interval of time between emission from C and from A is ℓ'/c. During this time the square moves by $\beta\ell'$, i.e. the length CA. Photons from A and B are emitted and received at the same time and therefore $AB = \ell'/\Gamma$. The total observed length is given by

$$CB = CA + AB = \frac{\ell'}{\Gamma}(1 + \Gamma\beta). \tag{1.9}$$

As β increases, the observer sees the side AB increasingly shortened by the Lorentz contraction, but at the same time the length of the side CA increases. The maximum total length is observed for $\beta = 1/\sqrt{2}$, corresponding to $\Gamma = \sqrt{2}$ and to $CB = \ell'\sqrt{2}$, i.e. equal to *the diagonal* of the square. Note that we have considered the square (and the bar in the previous section) to be at large distances from the observer, so that the emitted light rays are all parallel. If the object is near to the observer, we must take into account that different points of one side of the square (e.g. the side AB in Fig. 1.2) have different travel paths to reach the observer, producing additional distortions. See the book by Mook and Vargish (1991) for some interesting illustrations.
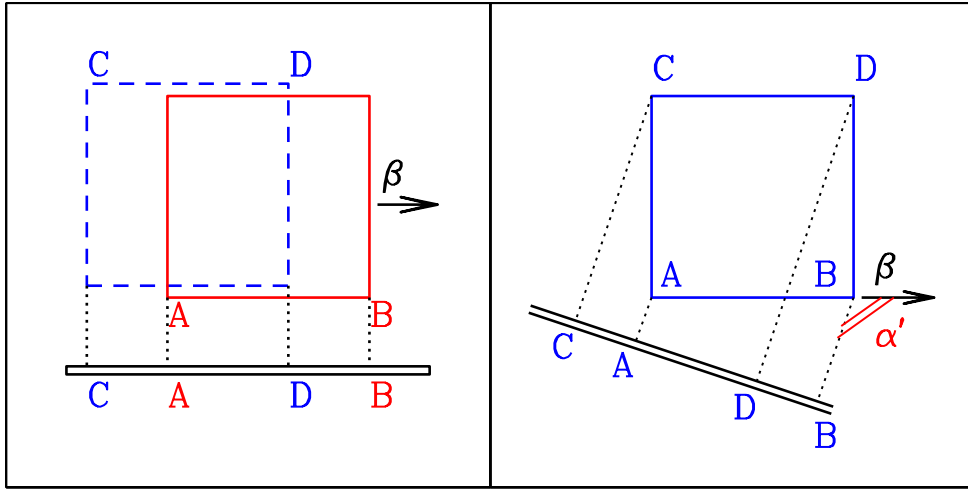
Figure 1.3: An observer that sees the object at rest at a viewing angle given by $\sin\alpha' = \delta\sin\alpha$, will take the same picture as the observer that sees the object moving and making an angle α with his/her line of sight. Note that $\sin\alpha' = \sin(2\pi - \alpha')$.

1.2.3 Rotation, not contraction

The net result (taking into account **both** the length contraction **and** the different paths) is an apparent **rotation** of the square, as shown in Fig. 1.2 (right panel). The rotation angle α can be simply derived (even geometrically) and is given by

$$\cos\alpha = \beta \tag{1.10}$$

A few considerations follow:

- If you rotate a sphere you still get a sphere: you **do not** observe a contracted sphere.

- The total length of the projected square, appearing on the film, is $\ell'(\beta + 1/\Gamma)$. It is maximum when the "rotation angle" $\alpha = 45° \rightarrow \beta = 1/\sqrt{2} \rightarrow \Gamma = \sqrt{2}$. This corresponds to the diagonal.

- The appearance of the square *is the same as what seen in a comoving frame for a line of sight making an angle α' with respect to the velocity vector, where α' is the aberrated angle* given by

$$\sin\alpha' = \frac{\sin\alpha}{\Gamma(1 - \beta\cos\alpha)} = \delta\sin\alpha \tag{1.11}$$

See Fig. 1.3 for a schematic illustration.
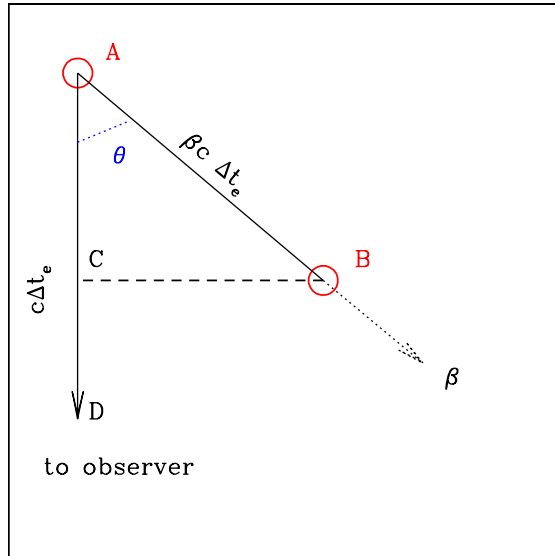
Figure 1.4: Difference between the proper time and the photons arrival time. A lamp, moving with a velocity βc, emits photons for a time interval $\Delta t'_e$ in its frame K'. The corresponding time interval measured by an observed at an angle θ, who receives the photons produced by the lamp is $\Delta t_a = \Delta t'_e/\delta$.

The last point is particularly important, because it introduces a great simplification in calculating not only the appearance of bodies with a complex shape but also the light curves of varying objects.

1.2.4   Time

Consider a lamp moving with velocity $v = \beta c$ at an angle θ from the line of sight. In K', the lamp remains on for a time $\Delta t'_e$. According to special relativity ("without photons") the measured time in frame K should be $\Delta t_e = \Gamma \Delta t'_e$ (time dilation). However, if we use photons to measure the time interval, we once again must consider that the first and the last photons have been emitted in different location, and their travel path lengths are different. In this case, to find out Δt_a, the time interval between the arrival of the first and last photon, consider Fig. 1.4. The first photon is emitted in A, the last in B. If these points are measured in frame K, then the path AB is

$$AB \,=\, \beta c \Delta t_e \,=\, \Gamma \beta c \Delta t'_e \tag{1.12}$$

While the lamp moved from A to B, the photon emitted when the lamp was in A has travelled a distance $AC = c\Delta t_e$, and is now in point D. Along the direction of the line of sight, the first and the last photons (the ones emitted in A and in B) are separated by CD. The corresponding time interval,

CD/c, is the interval of time Δt_{a} between the arrival of the first and the last photon:

$$
\begin{aligned}
\Delta t_{\mathrm{a}} &= \frac{CD}{c} = \frac{AD - AC}{c} = \Delta t_{\mathrm{e}} - \beta \Delta t_{\mathrm{e}} \cos \theta \\
&= \Delta t_{\mathrm{e}} (1 - \beta \cos \theta) \\
&= \Delta t'_{\mathrm{e}} \Gamma (1 - \beta \cos \theta) \\
&= \frac{\Delta t'_{\mathrm{e}}}{\delta}
\end{aligned}
\tag{1.13}
$$

If θ is small and the velocity is relativistic, then $\delta > 1$, and $\Delta t_{\mathrm{a}} < \Delta t_{\mathrm{s}}$, i.e. we measure a *time contraction* instead of time dilation. Note also that we recover the usual time dilation (i.e. $\Delta t_{\mathrm{a}} = \Gamma \Delta t'_{\mathrm{e}}$) if $\theta = 90°$, because in this case all photons have to travel the same distance to reach us.

Since a frequency is the inverse of time, it will transform as

$$
\nu = \nu' \delta
\tag{1.14}
$$

It is because of this that the factor δ is called the relativistic Doppler factor.

1.2.5 Aberration

Another very important effect happening when a source is moving is the aberration of light. It is rather simple to understand, if one looks at Fig. 1.5. A source of photons is located perpendilarly to the right wall of a lift. If the lift is not moving, and there is a hole in its right wall, then the ligth ray enters in A and ends its travel in B. If the lift is not moving, A and B are at the same heigth. If the lift is moving with a constant velocity v to the top, when the photon smashes the right wall it has a different location, and the point B will have, for a comoving observer, a smaller height than A. The light ray path now appears oblique, tilted. Of course, the greater v, the more tilted the light ray path appears. This immediately stimulate the question: what happens if the lift, instead to move with a constant velocity, is accelerating? This this example one can easily convince him/herself that the "trajectory" of the photon would appear curved. Since, by the equivalence principle, the accelerating lift cannot tell if there is an engine pulling him up or if there is a planet underneath it, we can then say that gravity bends the light rays, and make the space curved.

This helps to understand why angles, between two inertial frames, change. Calling θ the angle between the direction of the emitted photon and the source velocity vector, we have:

$$
\begin{aligned}
\sin \theta &= \frac{\sin \theta'}{\Gamma (1 + \beta \cos \theta')}; & \sin \theta' &= \frac{\sin \theta}{\Gamma (1 - \beta \cos \theta)} \\
\cos \theta &= \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}; & \cos \theta' &= \frac{\cos \theta - \beta}{1 - \beta \cos \theta}
\end{aligned}
\tag{1.15}
$$
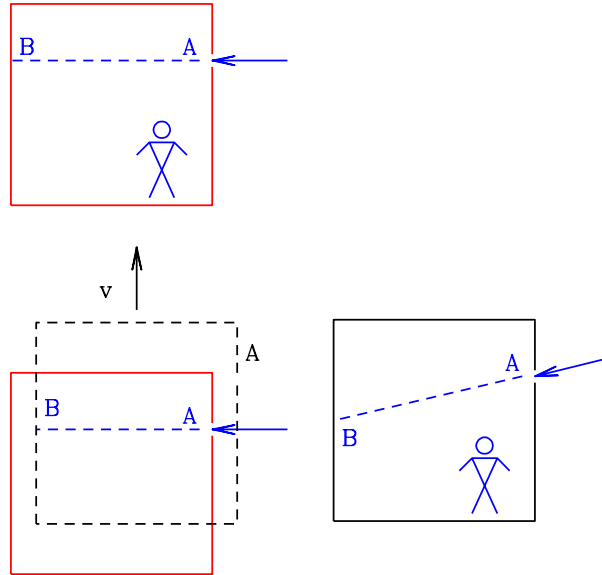
Figure 1.5: The relativistic lift, to explain relativistic aberration of light. Assume first a non–moving lift, with a hole on the right wall. A light ray, coming perpendiculrly to the left wall, enter through the wall in A and ends its travel in B. If the lift is moving with a constant velocity v to the top, its position is changed when the photon arrives to the left wall. For the comoving observer, therefore, it appears that the light path is tilted, since the point B where the photon smashes into the left wall is below the point A. What happens if the lift, instead to move with a constant velocity, is accelerating?

Note that, if $\theta' = 90°$, then $\sin\theta = 1/\Gamma$ and $\cos\theta = \beta$. Consider a souce emitting isotropically in K'. Halph of its photons are emitted in the emisphere, namely, with $\theta' \leq 90°$. Then, in K, the same source will appear to emit halph of its photons into a cone of semiaperture Γ.

Assuming symmetry around the angle ϕ, the transformation of the solid angle $d\Omega$ is

$$d\Omega \,=\, 2\pi d\cos\theta = \frac{d\Omega'}{\Gamma^2(1+\beta\cos\theta')^2} = d\Omega'\,\Gamma^2(1-\beta\cos\theta)^2 = \frac{d\Omega'}{\delta^2} \quad (1.16)$$

1.2.6   Intensity

We now have all the ingredients necessary to calculate the transformation of the specific (i.e. monochromatic) and bolometric intensity. The specific intensity has the unit of energy per unit surface, time, frequency and solid angle. In cgs, the units are [erg cm^{-2} s^{-1} Hz^{-1} ster^{-1}]. We can then write

the specific intensity as

$$
\begin{aligned}
I(\nu) &= h\nu \frac{dN}{dt\, d\nu\, d\Omega\, dA} \\
&= \delta h\nu' \frac{dN'}{(dt'/\delta)\, \delta d\nu'\, (d\Omega'/\delta^2)\, dA'} \\
&= \delta^3\, I'(\nu') = I'(\nu/\delta) \tag{1.17}
\end{aligned}
$$

Note that $dN = dN'$ because it is a number, and that $dA = dA'$ because it is an area perpendicular to the velocity. If we integrate over frequencies we obtain the bolometric intensity which transforms as

$$
I = \delta^4 I' \tag{1.18}
$$

The fourth power of δ can be understood in a simple way: one power comes from the transformation of the frequencies, one for the time, and two for the solid angle. They all add up. This transformation is at the base of our understanding of relativistic sources, namely radio–loud AGNs, gamma–ray bursts and galactic superluminal sources.

1.2.7 Luminosity and flux

The transformation of fluxes and luminosities from the comoving to the observer frames is not trivial. The most used formula is $L = \delta^4 L'$, but this assumes that we are dealing with a single, spherical blob. It can be simply derived by noting that $L = 4\pi d_{\rm L}^2 F$, where F is the observed flux, and by considering that the flux, for a distance source, is $F \propto \int_{\Omega_s} I d\Omega$. Since Ω_s is the source solid angle, which is the same in the two K and K' frames, we have that F transforms like I, and so does L. But the emission from jets may come not only by a single spherical blob, but by, for instance, many blobs, or even by a continuos distribution of emitting particles flowing in the jet. If we assume that the walls of the jet are fixed, then the concept of "comoving" frame is somewhat misleading, because if we are comoving with the flowing plasma, then we see the walls of the jet which are moving.

A further complication exists if the velocity is not uni–directional, but radial, like in gamma–ray bursts. In this case, assume that the plasma is contained in a conical narrow shell (width smaller than the distance of the shell from the apex of the cone). The observer which is moving together with a portion of the plasma, (the nearest case of a "comoving observer") will see the plasma close to her going away from her, and more so for more distant portions of the plasma. Indeed, there could be a limiting distance beyond which the two portions of the shells are causally disconnected.

Useful references are Lind & Blandford (1985) and Sikora et al. (1997).

The (frequency integrated) emissivity j is the energy emitted per unit time, solid angle and volume. We generally have that the intensity, for an

otpically thin source, is $I = \int_{\Delta R} j dr$, where ΔR is the length of the region containing the emitting particles. This quantity transforms like $j = j'\delta^3$, namely with one power of δ less than the intensity.
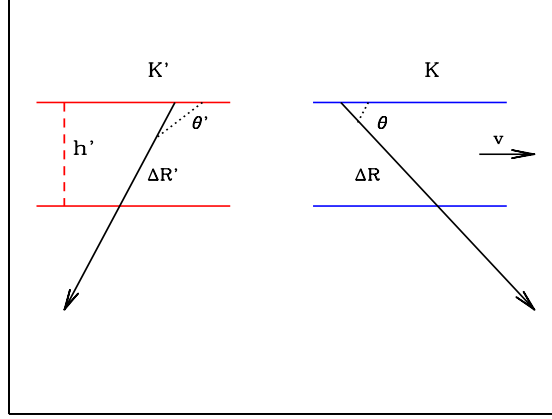
Figure 1.6: Due to aberration of light, the travel path of the a light ray is different in the two frames K and K'

To understand why, consider a slab with plasma flowing with a velocity parallel to the walls of the slab, as in Fig. 1.6. The observer in K will measure a certain ΔR which depends on her viewing angle. In K' the same path has a different length, because of the aberration of light. The height of the slab $h' = h$, since it is perpendicular to the velocity. The light ray travels a distance $\Delta R = h/\sin\theta$ in K, and the same light ray travels a distance $\Delta R' = h'/\sin\theta'$ in K'. Since $\sin\theta' = \sin\theta\delta$, then $\Delta R' = \delta R/\delta$. Therefore the column of plasma contributing to the emission, for $\delta > 1$, is less than what the observer in K would guess by measuring ΔR. For semplicity, assume that the plasma is homogenous, allowing to simply write $I = j\Delta R$. In this case:

$$I = j\Delta R = \delta^4 I' = \delta^4 j'\Delta R' \rightarrow j = \delta^3 j' \qquad (1.19)$$

And the corresponding transformation for the specific emissivity is $j(\nu) = \delta^2 j'(\nu')$.

Fig. 1.7 illustrates another interesting example, taken from the work of Sikora et al. (1997). Consider that within a distance R from the apex of a jet (R measured in K), at any given time there are N blobs (10 on the specific example of Fig. 1.7), moving with a velocity $v = \beta c$ along the jet. To fix the ideas, let assume that beyond R they switch off. If the viewing angle is $\theta = 90°$, the photons emitted by each blob travel the same distance to reach the observer, who will see all the 10 blobs. But if $\theta < 90°$, the photons produced by the rear blobs must travel for a longer distance in
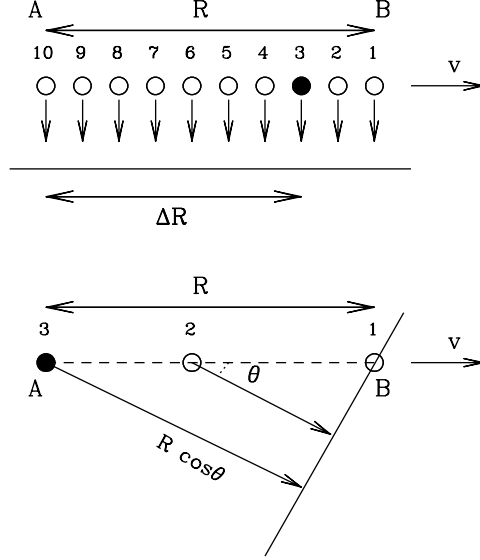
Figure 1.7: Due to the differences in light travel time, the number of blobs that can be observed simultaneously at any given time depends on the viewing angle and the velocity of the blobs. In the top panel the viewing angle is $\theta = 90°$ and all the blobs contained within a certain distance R can be seen. For smaller viewing angles, less blobs are seen. This is because the photons emitted by the rear blobs have more distance to travel, and therefore they have to be emitted before the photons produced by the front blob. Decreasing the viewing angle θ we see less blobs (3 for the case illustrated in the bottom panel).

order to reach the observer, and therefore they have to be emitted before the photons produced by the front blob. The observer will then see *less* blobs. To be more quantitative, consider a viewing angle $\theta < 90°$. Photons emitted by blob numer 3 to reach blobs number 1 when it produces its last photon (before to switch off) were emitted when the blobs itself was just born (it was crossing point A). They travelled a distance $R \cos\theta$ in a time Δt. During the same time, the blob number 3 travelled a distance $\Delta R = c\beta\Delta t$ in the forward direction. The fraction f of blobs that can be seen is then

$$f = \frac{R - \Delta R}{R} = 1 - \frac{c\beta\Delta t}{R} = 1 - \beta\cos\theta \qquad (1.20)$$

Where we have used the fact that $\Delta t = (R/c)\cos\theta$. This is the usual Doppler factor. We may multiply and divide by Γ to obtain

$$f = \frac{1}{\Gamma\delta} \qquad (1.21)$$

The bottom line is the following: even if the flux from a single blob is boosted by δ^4, if the jet is made by many (N) equal blobs, the total flux is not just boosted by $N\delta^4$ times the intrinsic flux of a blob, because the observer will see less blobs if $\theta < 90°$.

1.2.8   Brightness Temperature

The brightness temperature is a quantity used especially in radio astronomy, and it is defined by

$$T_{\rm B} \equiv \frac{I(\nu)}{2k}\,\frac{c^2}{\nu^2} \;=\; \frac{F(\nu)}{2\pi k\,\theta_{\rm s}^2}\,\frac{c^2}{\nu^2} \tag{1.22}$$

where we have assumed that the solid angle subtended by the source is $\Delta\Omega_{\rm s} \sim \pi\theta_{\rm s}^2$, and that the received flux is $F(\nu) = \Delta\Omega_{\rm s}I(\nu)$. There are 2 ways to measure $\theta_{\rm s}$:

1. from VLBI observations, one can often resolve the source and hence directly measure the angular size. In this case the relation between the brightness temperature measured in the K and K' frames is

$$T_{\rm B} \;=\; \frac{\delta^3 F'(\nu')}{2\pi k\,\theta_{\rm s}^2}\,\frac{c^2}{\delta^2(\nu')^2} \;=\; \delta\,T'_{\rm B} \tag{1.23}$$

2. If the source is varying, we can estimate its size by requiring that the observed variability time–scale $\Delta t_{\rm var}$ is longer than the light travel time R/c, where R is the typical radius of the emission region. In this case

$$T_{\rm B} \;>\; \frac{\delta^3 F'(\nu')}{2\pi k}\,\frac{d_{\rm A}^2\delta^2}{(c\Delta t'_{\rm var})^2}\,\frac{c^2}{\delta^2(\nu')^2} \;=\; \delta^3\,T'_{\rm B} \tag{1.24}$$

where $d_{\rm A}$ is the angular distance, related to the luminosity distance $d_{\rm L}$ by $d_{\rm A} = d_{\rm L}/(1+z)^2$.

There is a particular class of extragalactic radio sources, called Intra–Day Variable (IDV) sources, showing variability time–scales of hours in the radio band. For them, the corresponding observed brightess temperature can exceed 10^{16} K, a value much larger than the theoretical limit for an incoherent synchrotron source, which is between 10^{11} and 10^{12} K. If the variability is indeed intrinsic, namely not produced by interstellar scintillation, then one would derive a limit on the beamig factor δ, which should be larger than about 100.

1.2.9   Moving in an homogeneous radiation field

Jets in AGNs often moves in an external radiation field, produced by, e.g. the accretion disk, ot by the Broad Line Region (BLR) which intercept a

fraction of the radiation produved by the disk and re–emit it in the form of emission lines. It it therefore interesting to calculate what is the energy density seen by a an observer which is comoving with the jet plasma.
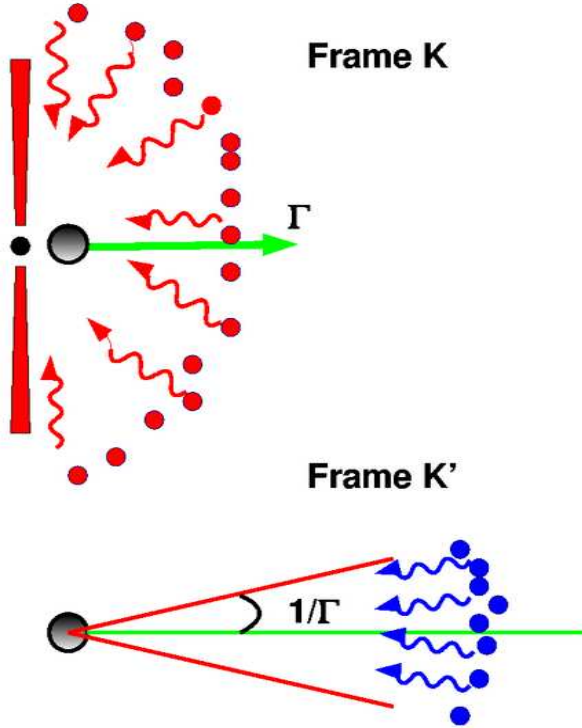
Figure 1.8: A real case: a relativistic bob is moving with the Broad Line Region of a radio loud AGN, with Lorentz factor Γ. In the rest frame K' of the blob the photons coming from $90°$ in frame K are seen to come at an angle $1/\Gamma$. The energy density as seen by the blob is enhanced by a factor $\sim \Gamma^2$.

To make a specific example, as illistrated bu Fig. 1.8, assume that a portion of the jet is moving with a bulk Lorentz factor Γ, velocity βc and that it is surrounded by an shell of broad line clouds. For simplicity, assume that the broad line photons are produced by the surface of a sphere of radius R and that the jet is within it. Assume also that the radiation is monochromatic at sum frequency ν_0 (in frame K). The comoving (in frame K') observer will see photons coming from an emisphere (the other half may be hidden by the accretion disk): photons coming from the forward direction are seen blue-shifted by a factor $(1+\beta)\Gamma$, while photons that the observer in K sees as coming from the side (i.e. $90°$ degrees) will be observed in K' as coming by an angle given by $\sin\theta' = 1/\Gamma$ (and $\cos\theta' = \beta$) and will be blue–shifted by a factor Γ. As seen in K', each element of the BLR surface is

moving in the opposite direction of the actual jet velocity, and the photons
emitted by this element form an angle θ' with respect the element velocity.
The Doppler factor used by K' is then

$$\delta' = \frac{1}{\Gamma(1 - \beta \cos \theta')} \tag{1.25}$$

The intensity coming from each element is seen boosted as (cfr Eq. 1.2.9):

$$I' = \delta'^4 I \tag{1.26}$$

The radiation energy density is the integral over the solid angle of the intensity, divided by c:

$$
\begin{aligned}
U' &= \frac{2\pi}{c} \int_\beta^1 I' d\cos\theta' \\
&= \frac{2\pi}{c} \int_\beta^1 \frac{I}{\Gamma^4(1 - \beta\cos\theta')^4} d\cos\theta' \\
&= \left(1 + \beta + \frac{\beta^2}{3}\right) \Gamma^2 \frac{2\pi I}{c} \\
&= \left(1 + \beta + \frac{\beta^2}{3}\right) \Gamma^2 U
\end{aligned}
\tag{1.27}
$$

Note that the limits of the integral correspond to the angles $0'$ and $90°$ in
frame K. The radiation energy density, in frame K', is then boosted by a
factor $(7/3)\Gamma^2$ when $\beta \sim 1$. Doing the same calculation for a sphere, one
would obtain $U' = \Gamma^2 U$.

Furthermore a (monochromatic) flux in K is seen, in K', at different
frequencies, between $\Gamma\nu_0$ and $(1 + \beta)\Gamma\nu_0$, with a slope $F'(\nu') \propto \nu'^2$. Why
the slope ν'^2? This can be derived as follows: we already know that $I'(\nu') =
\delta'^3 I(\nu) = (\nu'/\nu)^3 I(\nu)$. The flux at a specific frequency is

$$F'(\nu') = 2\pi \int_{\mu_1'}^{\mu_2'} d\mu' \left(\frac{\nu'}{\nu}\right)^3 I(\nu) \tag{1.28}$$

where $\mu' \equiv \cos\theta'$, and the integral is over those μ' contributing at ν'. Since

$$\frac{\nu'}{\nu} = \delta' = \frac{1}{\Gamma(1 - \beta\mu')} \rightarrow \mu' = \frac{1}{\beta}\left(1 - \frac{\nu}{\Gamma\nu'}\right) \tag{1.29}$$

we have

$$d\mu' = -\frac{d\nu}{\beta\Gamma\nu'} \tag{1.30}$$

Therefore, if the intensity is monochromatic in frame K, i.e. $I(\nu) = I_0\delta(\nu - \nu_0)$, the flux density in the comoving frame is

$$
\begin{aligned}
F'(\nu') &= 2\pi \int_{\nu_2}^{\nu_1} \frac{d\nu}{\beta\Gamma\nu'} \left(\frac{\nu'}{\nu}\right)^3 I_0\delta(\nu - \nu_0) \\
&= \frac{2\pi}{\Gamma\beta} \frac{I_0}{\nu_0^3} \nu'^2; \qquad \Gamma\nu_0 \leq \nu' \leq (1 + \beta)\Gamma\nu_0
\end{aligned}
\tag{1.31}
$$

where the frequency limits corresponds to photons produced in an emisphere in frame K, and between $0°$ and $\sin\theta' = 1/\Gamma$ in frame K'. Integrating Eq. 2.25 over frequency, one obtains

$$F' = 2\pi I_0 \Gamma^2 \left(1 + \beta + \frac{\beta^2}{3}\right) = \Gamma^2 \left(1 + \beta + \frac{\beta^2}{3}\right) F \qquad (1.32)$$

in agreement with Eq. 1.27.

$\nu = \nu'\delta$	frequency
$t = t'/\delta$	time
$V = V'\delta$	volume
$\sin\theta = \sin\theta'/\delta$	sine
$\cos\theta = (\cos\theta' + \beta)/(1 + \beta\cos\theta')$	cosine
$I(\nu) = \delta^3 I'(\nu')$	specific intensity
$I = \delta^4 I'$	total intensity
$j(\nu) = j'(\nu')\delta^2$	specific emissivity
$\kappa(\nu) = \kappa'(\nu')/\delta$	absorption coefficient
$T_B = T'_B\delta$	brightn. temp. (size directly measured)
$T_B = T'_B\delta^3$	brightn. temp. (size from variability)

Table 1.1: Useful relativistic transformations

1.3   A question

Suppose that some plasma of mass m is falling onto a central object with a velocity v and bulk Lorentz factor Γ. The central object has mass M and produces a luminosity L. Assume that the interaction is through Thomson scattering and that there are no electron–positron pairs.

   a) What is the radiation force acting on the electron?
   b) What is the gravity force acting on the proton?
   c) What definition of limiting ("Eddington") luminosity would you give in this case?
   d) What happens if the plasma is instead going outward?

References
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Chapter 2

Synchrotron emission and absorption

2.1 Introduction

We now know for sure that many astrophysical sources are magnetized and have relativistic leptons. Magnetic field and relativistic particles are the two ingredients to have synchrotron radiation. What is responsible for this kind of radiation is the Lorentz force, making the particle to gyrate around the magnetic field lines. Curiously enough, this force *does not work*, but makes the particles to accelerate even if their velocity modulus hardly changes.

The outline of this section is:

1. We will derive the total power emitted by the single electron. Total means integrated over frequency and over emission angles. This will require to generalize the Larmor formula to the relativistic case;

2. We will then outline the basics of the spectrum emitted by the single electron. This is treated is several text–books, so we will concentrate on the basic concepts;

3. Spectrum from an ensemble of electrons. Again, only the basics;

4. Synchrotron self absorption. We will try to discuss things from the point of view of a photon, that wants to calculate its survival probability, and also the point of view of the electron, that wants to calculate the probability to absorb the photon, and then increase its energy and momentum.

2.2 Total losses

To calculate the total (=integrated over frequencies and emission angles) synchrotron losses we go into the frame that is instantaneously at rest with

the particle (in this frame v is zero, but not the acceleration!). This is because we will use the fact that the *emitted* power is Lorentz invariant:

$$P_\mathrm{e} = P_\mathrm{e}' = \frac{2e^2}{3c^3}a'^2 = \frac{2e^2}{3c^3}\left[a_\parallel'^2 + a_\perp'^2\right] \tag{2.1}$$

where the subscript "e" stands for "emitted". The fact that the power is invariant sounds natural, since after all, power is energy over time, and both energy and time transforms the same way (in special relativity with rulers and clocks). But be aware that this *does not mean* that the emitted and *received* power are the same. They are not!

The problem is now to find how the parallel (to the velocity vector) and perpendicular components of the acceleration Lorentz transform. This is done in text books, so we report the results:

$$\begin{aligned} a_\parallel' &= \gamma^3 a_\parallel \\ a_\perp' &= \gamma^2 a_\perp \end{aligned} \tag{2.2}$$

where γ is the particle Lorentz factor. One easy way to understand and remember these transformations is to recall that the acceleration is the second derivative of space with respect to time. The perpendicular component of the displacement is invariant, so we have only to transform (twice) the time (factor γ^2). The parallel displacement instead transforms like γ, hence the γ^3 factor.

The generalization of the Larmor formula is then:

$$P_\mathrm{e} = P_\mathrm{e}' = \frac{2e^2}{3c^3}\left[a_\parallel'^2 + a_\perp'^2\right] = \frac{2e^2}{3c^3}\gamma^4\left[\gamma^2 a_\parallel^2 + a_\perp^2\right] \tag{2.3}$$

Don't be fooled by the γ^2 factor in front of $a_\parallel^2$... this component of the power is hardly important: since the velocity, for relativistic particles, is always close to c, it implies that one can get very very small acceleration in the same direction of the velocity. This is why linear accelerators minimize radiation losses. For synchrotron machines, instead, the losses due to radiation can be the limiting factor, and they are of course due to $a_\perp$: changing the *direction* of the velocity means large accelerations, even without any change in the velocity modulus. To go further, we have to calculate the two component of the acceleration for an electron moving in a magnetic field. Its trajectory, in general, will have an helical shape of radius r_L (the Larmor radius). The angle that the velocity vector makes with the magnetic field line is called *pitch angle*. Let us denote it with θ. We can anticipate that, in the absence of electric field and for a homogeneous magnetic field, the modulus of the velocity will not change: the magnetic field does not work, and so there is no change of energy, except for the losses due to the synchrotron radiation itself. So one important assumption is that *at least during one gyration, the*
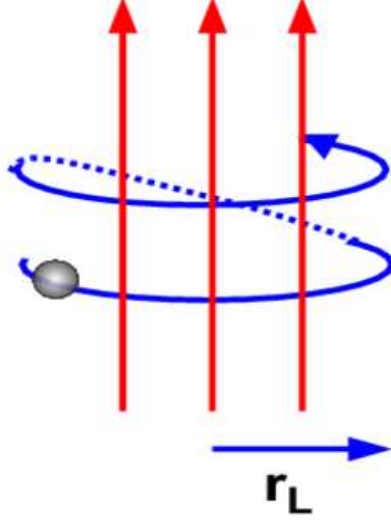
Figure 2.1: A particle gyrates along the magnetic field lines. Its trajectory has an helicoidal shape, with Larmor radius r_L and pitch angle θ.

losses are not important. This is almost always satisfied in astrophysical settings, but there are indeed some cases where this is not true.

When there is no electric field the only acting force is the (relativistic) Lorentz force:

$$F_L = \frac{d}{dt}(\gamma m\mathbf{v}) = \frac{e}{c}\mathbf{v} \times B \tag{2.4}$$

The parallel and perpendicular components are

$$F_{L\parallel} = e\, v_\parallel B = 0 \quad \rightarrow \quad a_\parallel = 0$$

$$F_{L\perp} = \gamma m\frac{dv_\perp}{dt} = e\,\frac{v_\perp}{c}B \quad \rightarrow \quad a_\perp = \frac{evB\sin\theta}{\gamma mc} \tag{2.5}$$

We can also derive the Larmor radius r_L by setting $a_\perp = v_\perp^2/r_L$, and so

$$r_L = \frac{v_\perp^2}{a_\perp} = \frac{\gamma mc^2\beta\sin\theta}{eB} \tag{2.6}$$

The fundamental frequency is the inverse of the time occurring to complete one orbit (gyration frequency), so $\nu_B = c\beta\sin\theta/(2\pi r_L)$, giving

$$\nu_B = \frac{eB}{2\pi\gamma mc} = \frac{\nu_L}{\gamma} \tag{2.7}$$

where $\nu_{\rm L}$ is the Larmor frequency, namely the gyration frequency for sub–relativistic particles. Larger B means smaller $r_{\rm L}$, hence greater gyration frequencies. Vice–versa, larger γ means larger inertia, thus larger $r_{\rm L}$, and smaller gyration frequencies. Substituting $a_\perp$ given in Eq. 2.5 in the generalized Larmor formula (Eq. 2.3) we get:

$$P_{\rm S} = \frac{2e^4}{3m^2c^3}B^2\gamma^2\beta^2\sin^2\theta \tag{2.8}$$

We can make it nicer (for future use) by recalling that:

- The magnetic energy density is $U_B \equiv B^2/(8\pi)$

- the quantity $e^2/(m_{\rm e}c^2)$, in the case of electrons, is the classical electron radius r_0

- the square of the electron radius is proportional to the Thomson scattering cross section $\sigma_{\rm T}$, i.e. $\sigma_{\rm T} = 8\pi r_0^2/3 = 6.65 \times 10^{-25}$ cm^2.

Making these substitutions, we have that the synchrotron power emitted by a single electron of given pitch angle is:

$$P_{\rm S}(\theta) = 2\sigma_{\rm T}cU_B\gamma^2\beta^2\sin^2\theta \tag{2.9}$$

In the case of an *isotropic distribution of pitch angles* we can average the term $\sin^2\theta$ over the solid angle. The result is 2/3, giving

$$\langle P_{\rm S} \rangle = \frac{4}{3}\sigma_{\rm T}cU_B\gamma^2\beta^2 \tag{2.10}$$

Now pause, and ask yourself:

- Is P_S valid only for relativistic particles, or does it describe correctly the radiative losses also for sub–relativistic ones?

- In the relativistic case the losses are proportional to the *square* of the electron energy. Do you understand why? And for sub–relativistic particles?

- What happens of we have protons, instead of electrons?

- What happens for $\theta \to 0$? Are you sure? (that losses vanishes..). Ok, but what happens to the *received* power when you have the lines of the magnetic field along the line of sight, and a beam of particles, all with a small pitch angles, shooting at you?

- Why on earth there is the scattering cross section? Is this a coincidence or does it hide a deeper fact?

2.2.1   Synchrotron cooling time

When you want to estimate a timescale of a quantity A, you can always write $t = A/\dot{A}$. In our case A is the energy of the particle. For electrons with an isotropic pitch angle distribution we have

$$t_{\mathrm{syn}} = \frac{E}{\langle P_{\mathrm{S}} \rangle} = \frac{\gamma m_{\mathrm{e}} c^2}{(4/3)\sigma_{\mathrm{T}} c U_B \gamma^2 \beta^2} \sim \frac{7.75 \times 10^8}{B^2 \gamma} \; \mathrm{s} = \frac{24.57}{B^2 \gamma} \; \mathrm{yr} \quad (2.11)$$

In the vicinity of a supermassive AGN black hole we can have $B = 10^3 B_3$ Gauss and $\gamma = 10^3 \gamma_3$, yielding $t_{\mathrm{syn}} = 0.75/(B_3^2 \gamma_3)$ s. The same electron, in the radio lobes of a radio loud quasars with $B = 10^{-5} B_{-5}$ Gauss, cools in $t_{\mathrm{syn}} = 246$ million years.

2.3   Spectrum emitted by the single electron

2.3.1   Basics

There exists a typical frequency associated to the synchrotron process. This is related to the inverse of a typical time. If the electron is relativistic, this is *not* the revolution period. Instead, it is the fraction of the time, for each orbit, during which the observer *receives* some radiation. To simplify, consider an electron with a pitch angle of $90°$, and look at Fig. 2.2, illustrating the typical patterns of the produced radiation for sub–relativistic electrons moving with a velocity parallel (top panel) or perpendicular (mid panel) to the acceleration. In the bottom panel we see the pattern for a relativistic electron (with $\mathbf{v} \perp \mathbf{a}$): it is strongly beamed in the forward direction. This is the direct consequence of the aberration of light, making half of the photons be emitted in a cone of semi–aperture angle $1/\gamma$ (which is called *the beaming angle*). Note that this *does not* mean that *half of the power* is emitted within $1/\gamma$, because the photons inside the beaming cone are more energetic than those outside, and are more tightly packed (do you remember the δ^4 factor when studying beaming?).

To go further, recall what we do when we study a time series and we want to find the power spectrum: we Fourier transform it. In this case we must do the same. Therefore if there is a typical timescale during which we *receive* most of the signal, we can say that most of the power is emitted at a frequency that is the inverse of that time.

Look at Fig. 2.3: the relativistic electron emits photons all along its orbit, but it will "shoot" in a particular direction only for the time

$$\Delta t_{\mathrm{e}} \sim \frac{AB}{v} = \frac{2\pi r_L}{2\gamma v} = \frac{2}{\gamma \nu_{\mathrm{B}}} \quad (2.12)$$

where we made use of the the definition $\nu_{\mathrm{B}} \equiv v/(2\pi r_{\mathrm{L}})$. This is the *emitting* time during which the electron emits photons that will reach the observer.
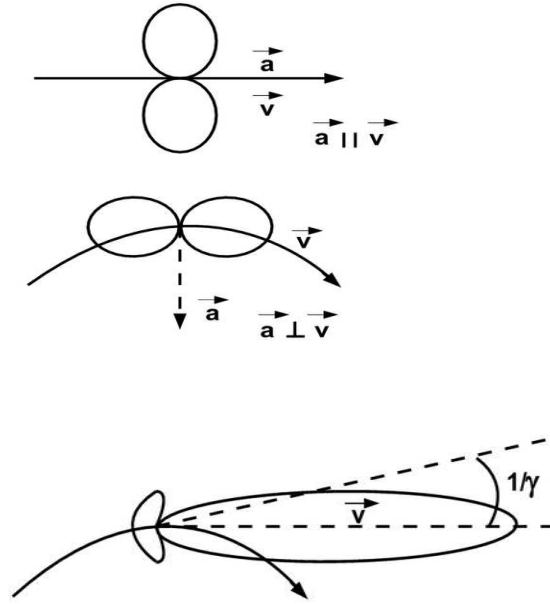
Figure 2.2: Radiation patterns for a non relativistic particle with the velocity parallel (top) or perpendicular (mid) to the acceleration. When the particle is relativistic, the pattern strongly changes due to the aberration of light, and is strongly beamed in the forward direction.

We can approximate the arc AB with a straight segment if the electron is relativistic, and the observed will then measure an *arrival* time Δt_A that is shorter than Δt_e:

$$\Delta t_A = \Delta t_e \, (1 - \beta) = \Delta t_e \, \frac{(1 - \beta^2)}{1 + \beta} \sim \frac{\Delta t_e}{2\gamma^2} = \frac{1}{\gamma^3 \nu_B} \tag{2.13}$$

The inverse of this time is the typical synchrotron frequency:

$$\nu_s = \frac{1}{\Delta t_A} = \gamma^3 \nu_B = \gamma^2 \, \frac{eB}{2\pi m_e c} \tag{2.14}$$

This is a factor γ^3 greater than the fundamental frequency, and a factor γ^2 greater than the Larmor frequency, defined as the typical frequency of non–relativistic particles. We expect that the particle emits most of its power at this frequency.

2.3.2   The real stuff

One can look at any text book for a detailed discussion of the procedure to calculate the spectrum emitted by the single particle. Here we report
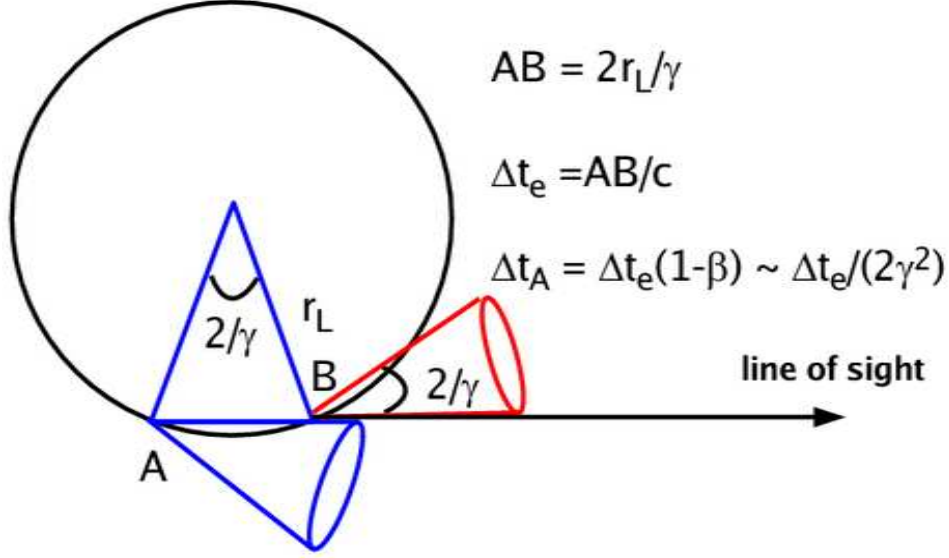
Figure 2.3: A relativistic electron is gyrating along a magnetic field line with pitch angle 90°. Its trajectory is then a circle of radius r_L. Due to aberration, an observer will "see it" (i.e. will measure an electric field) when the beaming cone of total aperture angle $2/\gamma$ is pointing at him.

the results: the power per unit frequency emitted by an electron of given Lorentz factor and pitch angle is:

$$
\begin{aligned}
P_s(\nu, \gamma, \theta) &= \frac{\sqrt{3}e^3 B \sin\theta}{m_e c^2} F(\nu/\nu_c) \\
F(\nu/\nu_c) &\equiv \frac{\nu}{\nu_c} \int_{\nu/\nu_c}^{\infty} K_{5/3}(y) dy \\
\nu_c &\equiv \frac{3}{2} \nu_s \sin\theta
\end{aligned}
\tag{2.15}
$$

This is the power integrated over the emission pattern. $K_{5/3}(y)$ is the modified Bessel function of order 5/3. The dependence upon frequency is contained in $F(\nu/\nu_c)$, that is plotted in Fig. 2.4. This function peaks at $\nu \sim 0.29\nu_c$, therefore very close to what we have estimated before, in our very approximate treatment. The low frequency part is well approximated by a power law of slope 1/3:

$$
F(\nu/\nu_c) \rightarrow \frac{4\pi}{\sqrt{3}\Gamma(1/3)} \left(\frac{\nu}{2\nu_c}\right)^{1/3} \quad (\nu \ll \nu_c)
\tag{2.16}
$$

At $\nu \gg \nu_c$ the function decays exponentially, and can be approximated by:

$$
F(\nu/\nu_c) \rightarrow \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{\nu}{\nu_c}\right)^{1/2} e^{-\nu/\nu_c} \quad (\nu \gg \nu_c)
\tag{2.17}
$$

Figure 2.4: Top panel: The function $F(\nu/\nu_c)$ describing the synchrotron spectrum emitted by the single electron. Bottom panel: $F(\nu/\nu_c)$ is compared with some approximating formulae, as labeled. We have defined $x \equiv \nu/\nu_c$.

Another approximation valid for all frequency, but overestimating F around the peak, is:

$$F(\nu/\nu_c) \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)} \left(\frac{\nu}{2\nu_c}\right)^{1/3} e^{-\nu/\nu_c} \qquad (2.18)$$

2.3.3 Limits of validity

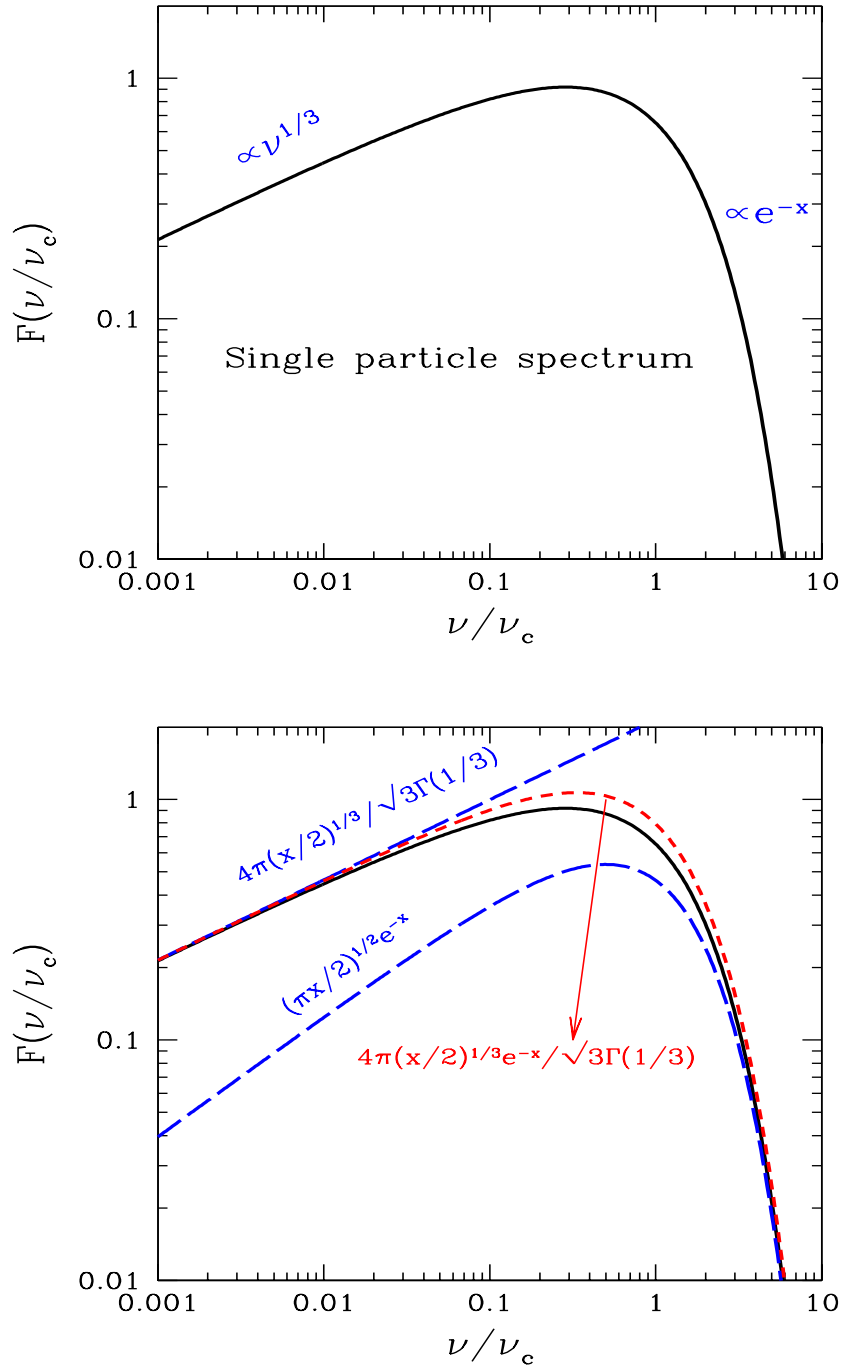One limit can be obtained by requiring that, during one orbit, the emitted energy is much smaller than the electron energy. If not, the orbit is modified, and our calculations are no more valid. For non–relativistic electrons this translates in demanding that

$$h\nu_{\mathrm{B}} < m_{\mathrm{e}}c^2 \rightarrow B < \frac{2\pi m_{\mathrm{e}}^2 c^3}{he} \equiv B_{\mathrm{c}} \qquad (2.19)$$

where $B_{\mathrm{c}} \sim 4.4 \times 10^{13}$ Gauss is the critical magnetic field, around and above which quantum effects appears (i.e. quantized orbits, Landau levels and so on).

For relativistic particles we demand that the energy emitted during one orbit does not exceed the energy of the particle.

$$\frac{P_{\mathrm{s}}}{\nu_{\mathrm{B}}} < \gamma m_{\mathrm{e}}c^2 \rightarrow B < \frac{e/\sigma_{\mathrm{T}}}{\gamma^2 \sin^2\theta} \sim \frac{7.22 \times 10^{14}}{\gamma^2 \sin^2\theta} \; \text{Gauss} \qquad (2.20)$$

Therefore for large γ we reach the validity limit even if the magnetic field is sub–critical.

For very small pitch angles beware that the spectrum is not described by $F(\nu/\nu_c)$, but consists of a blue-shifted cyclotron line. This is because, in the gyroframe, the particle is sub–relativistic, and so it emits only one (or very few) harmonics, that the observer sees blueshifted.

2.3.4 From cyclotron to synchrotron emission

A look at Fig. 2.5 helps to understand the difference between cyclotron and synchrotron emission. When the particle is very sub–relativistic, the observed electric field is sinusoidal in time. Correspondingly, the Fourier transform of $E(t)$ gives only one frequency, the first harmonic. Increasing somewhat the velocity (say, $\beta \sim 0.01$) the emission pattern starts to be asymmetric (for light aberration) and as a consequence $E(t)$ must be described by more than just one sinusoid, and higher order harmonics appear. In these cases the ratio of the power contained in successive harmonics goes as β^2.

Finally, for relativistic (i.e. $\gamma \gg 1$) particles, the pattern is so asymmetric that the observers sees only spikes of electric field. They repeat themselves with the gyration period, but all the power is concentrated into Δt_{A}. To reproduce $E(t)$ in this case with sinusoids requires a large number

of them, with frequencies going at least up to $1/\Delta t_A$. In this case the harmonics are many, guaranteeing that the spectrum becomes continuous with any reasonable line broadening effect, and the power is concentrated at high frequencies.

2.4   Emission from many electrons

Again, this problem is treated in several text books, so we repeat the basic results using some approximations, tricks and shortcuts.

The queen of the particle energy distributions in high energy astrophysics is the *power law* distribution:

$$N(\gamma) = K\,\gamma^{-p} = N(E)\,\frac{dE}{d\gamma}; \qquad \gamma_{\min}\gamma < \gamma_{\max} \tag{2.21}$$

Now, assuming that the distribution of pitch angles is the same at low and high γ, we want to obtain the synchrotron emissivity produced by these particles. Beware that the *emissivity* is the power per unit *solid angle* produced within 1 cm^3. The specific emissivity is also per unit of frequency. So, if Eq. 2.21 represents a density, we should integrate over γ the power produced by the single electron (of a given γ) times $N(\gamma)$, and divide all it by 4π, if the emission is isotropic:

$$\epsilon_s(\nu,\theta) = \frac{1}{4\pi}\int_{\gamma_{\min}}^{\gamma_{\max}} N(\gamma)P(\gamma,\nu,\theta)d\gamma \tag{2.22}$$

Doing the integral one easily finds that, in an appropriate range of frequencies:

$$\epsilon_s(\nu,\theta) \propto KB^{(p+1)/2}\nu^{-(p-1)/2} \tag{2.23}$$

The important thing is that a power law electron distribution produces a power law spectrum, and the two spectral indices are related. We traditionally call α the spectral index of the radiation, namely $\epsilon_s \propto \nu^{-\alpha}$. We then have

$$\alpha = \frac{p-1}{2} \tag{2.24}$$

This result is so important that it is worth to try to derive it in a way as simple as possible, even without doing the integral of Eq. 2.22. We can in fact use the fact that the synchrotron spectrum emitted by the single particle is peaked. We can then say, without being badly wrong, that all the power is emitted at the typical synchrotron frequency:

$$\nu_s = \gamma^2\nu_L; \qquad \nu_L \equiv \frac{eB}{2\pi m_e c} \tag{2.25}$$

In other words, there is a tight correspondence between the energy of the electron and the frequency it emits. To simplify further, let us assume that
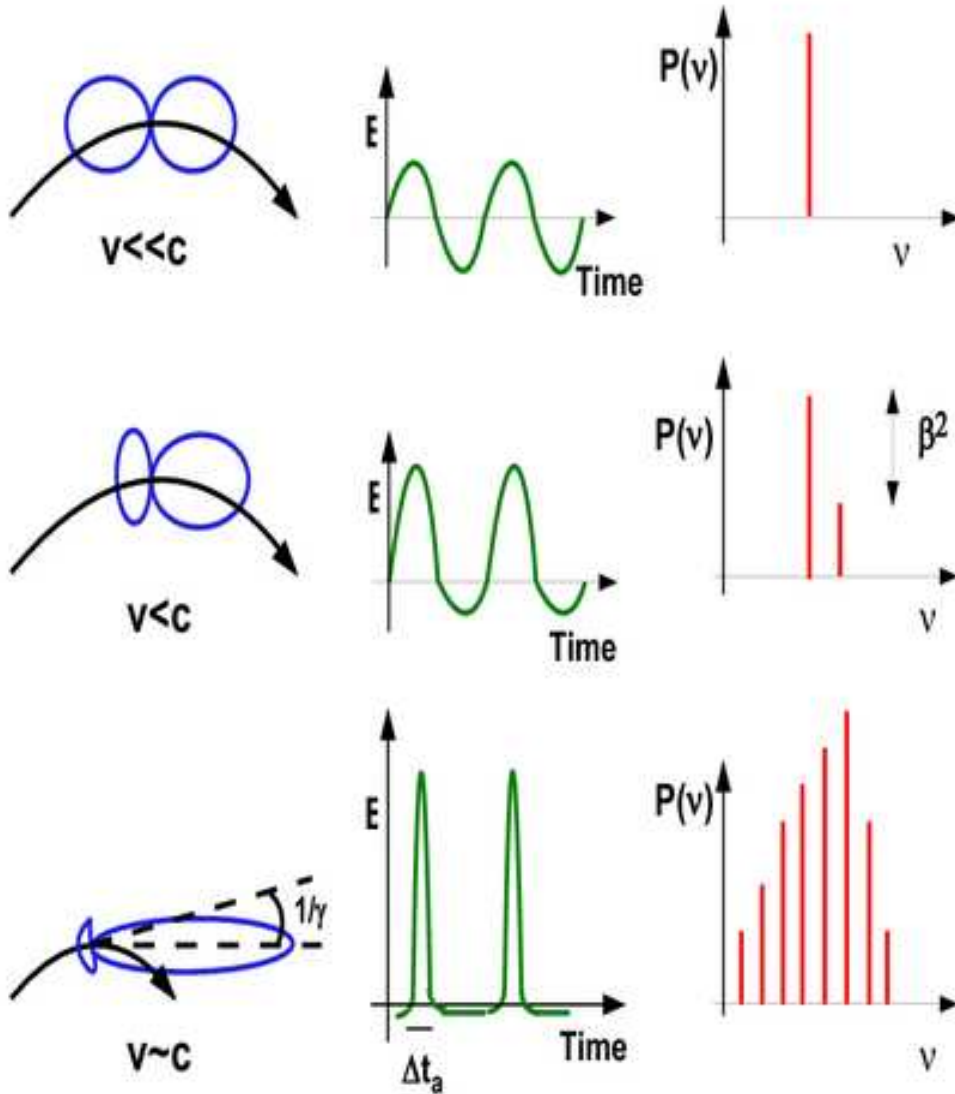
Figure 2.5: From cyclo to synchro: if the emitting particle has a very small velocity, the observer sees a sinusoidal (in time) electric field $E(t)$. Increasing the velocity the pattern becomes asymmetric, and the second harmonic appears. For $0 < \beta \ll 1$ the power in the second harmonic is a factor β^2 less than the power in the first. For relativistic particles, the pattern becomes strongly beamed, the emission is concentrated in the time Δt_A. As a consequence the Fourier transformation of $E(t)$ must contain many harmonics, and the power is concentrated in the harmonics of frequencies $\nu \sim 1/\Delta t_A$. Broadening of the harmonics due to several effects ensures that the spectrum in this case becomes continuous. Note that the fundamental harmonic becomes *smaller* increasing γ (since $\nu_B \propto 1/\gamma$).

the pitch angle is $90°$. The emissivity at a given frequency, within an interval $d\nu$, is then the result of the emission of electrons having the appropriate energy γ, within the interval $d\gamma$

$$\epsilon_{\mathrm{s}}(\nu)d\nu \; = \; \frac{1}{4\pi}\,P_{\mathrm{s}}N(\gamma)d\gamma; \quad \gamma = \left(\frac{\nu}{\nu_{\mathrm{L}}}\right)^{1/2}; \quad \frac{d\gamma}{d\nu} = \frac{\nu^{-1/2}}{2\nu_{\mathrm{L}}^{1/2}} \qquad (2.26)$$

we then have

$$\begin{aligned}
\epsilon_{\mathrm{s}}(\nu) &\propto& B^2\gamma^2 K\gamma^{-p}\frac{d\gamma}{d\nu} \\
&\propto& B^2 K \left(\frac{\nu}{\nu_{\mathrm{L}}}\right)^{(2-p)/2}\frac{\nu^{-1/2}}{\nu_{\mathrm{L}}^{1/2}} \\
&\propto& K\,B^{(p+1)/2}\,\nu^{-(p-1)/2} \qquad (2.27)
\end{aligned}$$

where we have used $\nu_{\mathrm{L}} \propto B$.

The synchrotron flux received from a homogeneous and thin source of volume $V \propto R^3$, at a distance d_{L}, is

$$\begin{aligned}
F_{\mathrm{s}}(\nu) &=& 4\pi\epsilon_{\mathrm{s}}(\nu)\frac{V}{4\pi d_{\mathrm{L}}^2} \\
&\propto& \frac{R^3}{d_{\mathrm{L}}^2}\,KB^{1+\alpha}\nu^{-\alpha} \\
&\propto& \theta_s^2 RKB^{1+\alpha}\nu^{-\alpha} \qquad (2.28)
\end{aligned}$$

where θ_s is the angular radius of the source (not the pitch angle!). Observing the source at two different frequencies allows to determine α, hence the slope of the particle energy distribution. Furthermore, if we know the distance and R, the normalization depends on the particle density and the magnetic field: two unknowns and only one equation. We need another relation to close the system. As we will see in the following, this is provided by the self–absorbed flux.

2.5   Synchrotron absorption: photons

All emission processes have their absorption counterpart, and the synchrotron emission is no exception. What makes synchrotron special is really the fact that it is done by relativistic particles, and they are almost never distributed in energy as a Maxwellian. If they were, we could use the well known fact that the ratio between the emissivity and the absorption coefficient is equal to the black body (Kirchhoff law) and then we could easily find the absorption coefficient. But in the case of a non–thermal particle distribution we cannot do that. Instead we are obliged to go back to more fundamental

relations, the one between the A and B Einstein coefficients relating spontaneous and stimulated emission and "true" absorption (by the way, recall that the absorption coefficient is what remains subtracting stimulated emission from "true" absorption). But we once again will use some tricks, in order to be as simple as possible. These are the steps:

1. The first trick is to think to our power law energy distribution as a superposition of Maxwellians, of different temperatures. So, we will relate the energy $\gamma m_e c^2$ of a given electron to the energy kT of a Maxwellian.

2. We have already seen that there is a tight relation between the emitted frequency and γ. Since the emission and absorption processes are related, we will assume that a particular frequency ν is preferentially absorbed by those electrons that can emit it.

3. As a consequence, we can associate our "fake" temperature to the frequency:

$$kT \sim \gamma m_e c^2 \sim m_e c^2 \left(\frac{\nu}{\nu_L} \right)^{1/2} \tag{2.29}$$

4. For an absorbed source the *brightness temperature* T_b, defined by

$$I(\nu) \equiv 2kT_b \frac{\nu^2}{c^2} \tag{2.30}$$

must be equal to the kinetic "temperature" of the electrons, and so

$$
\begin{aligned}
I(\nu) &\equiv 2kT \frac{\nu^2}{c^2} \sim 2m_e \nu^2 \left(\frac{\nu}{\nu_L} \right)^{1/2} \\
&\propto \frac{\nu^{5/2}}{B^{1/2}}
\end{aligned}
\tag{2.31}
$$

These are the right dependencies. Note that the spectrum is $\propto \nu^{5/2}$, not ν^2, and this is the consequence of having "different temperatures". Note also that the particle density disappeared: if you think about it is natural: the more electrons you have, the more you emit, but the more you absorb. Finally, even the *slope* of the particle distribution is not important, it controls (up to a factor of order unity) only the normalization of $I(\nu)$ (our ultra–simple derivation cannot account for it, see the Appendix).

The above is valid as long as we can associate a specific γ to any ν. This is not always the case. Think for instance to a cut–off distribution, with $\gamma_{\min} \gg 1$. In this case the electrons with $\gamma_{\min}$ are the most efficient emitters and absorbers of all photons with $\nu < \nu_{\min} \equiv \gamma_{\min}^2 \nu_L$. So in this case we *should not* associate a different temperature when dealing with different
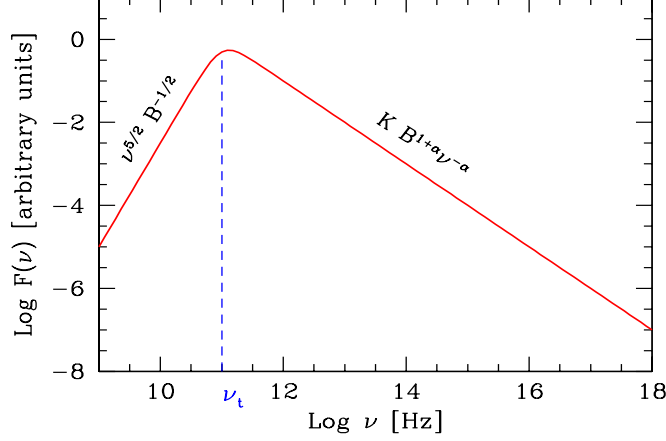
Figure 2.6: The synchrotron spectrum from a partially self absorbed source. Observations of the self absorbed part could determine B. Observations of the thin part can then determine K and the electron slope p.

$\nu < \nu_{\min}$. But if do not change T, we recover a self–absorbed intensity $I(\nu) \propto \nu^2$ (i.e. Raleigh–Jeans like).

Now, going from the intensity to the flux, we must integrate $I(\nu)$ over the angular dimension of the source (i.e. θ_s), obtaining

$$F(\nu) \propto \theta_s^2 \frac{\nu^{5/2}}{B^{1/2}} \tag{2.32}$$

if we could observe a self–absorbed source, of known angular size, we could then derive its magnetic field *even without knowing its distance.*

2.5.1   From thick to thin

To describe the transition from the self absorbed to the thin regime we have to write the radiation transfer equation. The easiest one is for a slab. Calling κ_ν the specific absorption coefficient [cm^{-1}] we have

$$I(\nu) = \frac{\epsilon(\nu)}{\kappa_\nu}(1 - e^{-\tau_\nu}); \quad \tau_\nu \equiv R\kappa_\nu \tag{2.33}$$

it is instructive to write Eq. 2.33 in the form:

$$I(\nu) = \epsilon(\nu)R\,\frac{1 - e^{-\tau_\nu}}{\tau_\nu} \tag{2.34}$$

because in this way it is evident that when $\tau_\nu \gg 1$ (self absorbed regime), we simply have

$$I(\nu) = \frac{\epsilon(\nu)R}{\tau_\nu} = \frac{\epsilon(\nu)}{\kappa_\nu}; \quad \tau_\nu \gg 1 \tag{2.35}$$
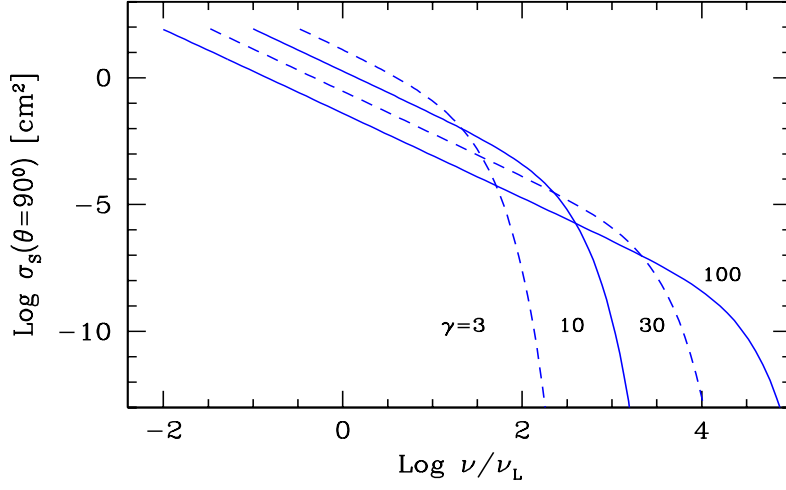
Figure 2.7: The synchrotron absorption cross section as a function of ν/ν_L for different values of γ, as labeled, assuming a pitch angle of $\theta = 90°$ and a magnetic field of 1 Gauss.

One can interpret it saying that the intensity is coming from electrons lying in a region R/τ_ν.

Since we have already obtained $I(\nu) \propto \nu^{5/2}B^{-1/2}$ in the absorbed regime, we can derive the dependencies of the absorption coefficient:

$$\kappa_\nu = \frac{\epsilon(\nu)}{I(\nu)} \propto \frac{KB^{(p+1)/2}\nu^{-(p-1)/1}}{\nu^{5/2}B^{-1/2}} = KB^{(p+2)/2}\nu^{-(p+4)/2} \qquad (2.36)$$

Note the rather strong dependence upon frequency: at large frequencies, absorption is small.

The obvious division between the thick and thin regime is when $\tau_\nu = 1$. We call *self–absorption frequency*, ν_t, the frequency when this occurs. We then have:

$$\tau_{\nu_t} = R\kappa_{\nu_t} = 1 \rightarrow \nu_t \propto \left[RK\,B^{(p+2)/2}\right]^{2/(p+4)} \qquad (2.37)$$

The self–absorption frequency is a crucial quantity for studying synchrotron sources: part of the reason is that it can be thought to belong to both regimes (thin and thick), the other reason is that the synchrotron spectrum peaks very close to ν_t (see Fig. 2.6) even if not exactly at ν_t (see the Appendix).

2.6 Synchrotron absorption: electrons

In the previous section we have considered what happens to the emitted spectrum when photons are emitted and absorbed. This is described by

the absorption coefficient. But now imagine to be an electron, that emits and absorbs synchrotron photons. You would probably be interested if your budget is positive or negative, that is, if you are loosing or gaining energy. This is most efficiently described by a cross section, that tells you the probability to absorb a photon. Surprisingly, the synchrotron absorption cross section has been derived relatively recently (Ghisellini and Svensson 1991), and its expression is:

$$\sigma_{\rm s}(\nu, \gamma, \theta) = \frac{16\pi^2}{3\sqrt{3}} \frac{e}{B} \frac{1}{\gamma^5 \sin\theta} K_{5/3}\left(\frac{\nu}{\nu_{\rm c} \sin\theta}\right) \qquad (2.38)$$

For frequencies $\nu \ll \nu_{\rm c}$ this expression can be approximated by:

$$\sigma_{\rm s}(\nu, \gamma, \theta) = \frac{8\pi^2 (3\sin\theta)^{2/3}\Gamma(5/3)}{3\sqrt{3}} \frac{e}{B} \left(\frac{\nu}{\nu_{\rm L}/\gamma}\right)^{-5/3}; \quad \nu \ll \nu_{\rm c} \qquad (2.39)$$

Note these features:

- At the fundamental frequency $\nu_{\rm L}/\gamma$, the cross section does not depend on γ.

- The dimensions are given by e/B: this factor is proportional to the product of the classical electron radius and the Larmor wavelength (or radius). Imagine an electron with $90°$ pitch angle, and to see its orbit from the side: you would see a rectangle of base $r_{\rm L}$ and height r_0. The area of this rectangle is of the order of e/B. At low frequencies, $\sigma_{\rm s}$ can be orders of magnitudes larger than the Thomson scattering cross section.

- There is no explicit dependence on the particle mass. However, protons have much smaller $\nu_{\rm L}$, and the dependence on mass is hidden there. Nevertheless, electrons and protons have the same cross section (of order e/B) at their respective fundamental frequencies.

Fig. 2.7 shows $\sigma_{\rm s}$ as a function of $\nu/\nu_{\rm L}$ for different γ. The thing it should be noticed is that this cross section is really large. Can we make some useful use of it? Well, there are at least two issues, one concerning energy, and the other concerning momentum.

First, electrons emitting and absorbing synchrotron photons do so with a large efficiency. *They can talk each other by exchanging photons.* Therefor, even if they are distributed as a power law in energy at the beginning, they will try to form a Maxwellian. They will form it, as long as other competing processes are not important, such as inverse Compton scatterings. The formation of the Maxwellian will interest only the low energy part of the electron distribution, where absorption is important. Note that this thermalization process works exactly when Coulomb collisions fail: they are

inefficient at low density and high temperature, while synchrotron absorption can work for relativistic electrons even if they are not very dense.

The second issue concerns exchange of momentum between photons and electrons. Suppose that a magnetized region with relativistic electrons is illuminated by low frequency radiation by another source, located aside. The electrons will efficiently absorb this radiation, and thus its momentum. The magnetized region will then accelerate.

References

Ghisellini G. & Svensson R., 1991, MNRAS, 252, 313
Ghisellini G., Haardt F. & Svensson R., 1998, MNRAS, 297, 348

Chapter 3

Appendix: Useful Formulae

In this section we collect several useful formulae concerning the synchrotron emission. When possible, we give also simplified analytical expressions. We will often consider that the emitting electrons have a distribution in energy which is a power law between some limits γ_1 and γ_2. Electrons are assumed to be isotropically distributed in the comoving frame of the emitting source. Their density is

$$N(\gamma) = K\gamma^{-p}; \quad \gamma_1 < \gamma < \gamma_2 \tag{3.1}$$

The Larmor frequency is defined as:

$$\nu_L \equiv \frac{eB}{2\pi m_{\rm e}c} \tag{3.2}$$

3.1 Synchrotron

3.1.1 Emissivity

The synchrotron emissivity $\epsilon_{\rm s}(\nu, \theta)$ [erg cm^{-3} s^{-1} sterad^{-1}] is

$$\epsilon_{\rm s}(\nu, \theta) \equiv \frac{1}{4\pi} \int_{\gamma_1}^{\gamma_2} N(\gamma) P_{\rm s}(\nu, \gamma, \theta) d\gamma \tag{3.3}$$

where $P_{\rm s}(\nu, \gamma, \theta)$ is the power emitted at the frequency ν (integrated over all directions) by the single electron of energy $\gamma m_{\rm e}c^2$ and pitch angle θ. For electrons making the same pitch angle θ with the magnetic field, the emissivity is

$$\epsilon_{\rm s}(\nu, \theta) = \frac{3\sigma_{\rm T}cKU_B}{8\pi^2\nu_L} \left(\frac{\nu}{\nu_L}\right)^{-\frac{p-1}{2}} (\sin\theta)^{\frac{p+1}{2}} 3^{\frac{p}{2}} \frac{\Gamma\left(\frac{3p-1}{12}\right)\Gamma\left(\frac{3p+19}{12}\right)}{p+1} \tag{3.4}$$

between $\nu_1 \gg \gamma_1^2\nu_L$ and $\nu_2 \ll \gamma_2^2\nu_L$. If the distribution of pitch angles is isotropic, we must average the $(\sin\theta)^{\frac{p+1}{2}}$ term, obtaining

$$< (\sin\theta)^{\frac{p+1}{2}} > = \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{p+1}{2}} \sin\theta d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+5}{4}\right)}{\Gamma\left(\frac{p+7}{4}\right)} \qquad (3.5)$$

Therefore the pitch angle averaged synchrotron emissivity is

$$\epsilon_{\rm s}(\nu) = \frac{3\sigma_{\rm T}cKU_B}{16\pi\sqrt{\pi}\nu_L} \left(\frac{\nu}{\nu_L}\right)^{-\frac{p-1}{2}} f_\epsilon(p) \qquad (3.6)$$

The function $f_\epsilon(p)$ includes all the products of the Γ–functions:

$$\begin{aligned}
f_\epsilon(p) &= \frac{3^{\frac{p}{2}}}{p+1} \frac{\Gamma\left(\frac{3p-1}{12}\right)\Gamma\left(\frac{3p+19}{12}\right)\Gamma\left(\frac{p+5}{4}\right)}{\Gamma\left(\frac{p+7}{4}\right)} \\
&\sim 3^{\frac{p}{2}} \left(\frac{2.25}{p^{2.2}} + 0.105\right) \qquad (3.7)
\end{aligned}$$

where the simplified fitting function is accurate at the per cent level.

3.1.2   Absorption coefficient

The absorption coefficient $\kappa_\nu(\theta)$ [cm^{-1}] is defined as:

$$\kappa_\nu(\theta) \equiv \frac{1}{8\pi m_{\rm e}\nu^2} \int_{\gamma_1}^{\gamma_2} \frac{N(\gamma)}{\gamma^2} \frac{d}{d\gamma}\left[\gamma^2 P(\nu,\theta)\right] d\gamma \qquad (3.8)$$

Written in this way, the above formula is valid even when the electron distribution is truncated. For our power law electron distribution $\kappa_\nu(\theta)$ becomes:

$$\kappa_\nu(\theta) \equiv \frac{1}{8\pi m_{\rm e}\nu^2} \int_{\gamma_1}^{\gamma_2} \frac{N(\gamma)}{\gamma^2} \frac{d}{d\gamma}\left[\gamma^2 P(\nu,\theta)\right] d\gamma \qquad (3.9)$$

Above $\nu = \gamma_1^2\nu_L$, we have:

$$\kappa_\nu(\theta) = \frac{e^2 K}{4m_{\rm e}c^2} (\nu_L \sin\theta)^{\frac{p+2}{2}} \nu^{-\frac{p+4}{2}} 3^{\frac{p+1}{2}} \Gamma\left(\frac{3p+22}{12}\right) \Gamma\left(\frac{3p+2}{12}\right) \quad (3.10)$$

Averaging over the pitch angles we have:

$$< (\sin\theta)^{\frac{p+2}{2}} > = \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{p+2}{2}} \sin\theta d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+8}{4}\right)} \qquad (3.11)$$

resulting in a pitch angle average absorption coefficient:

$$\kappa_\nu = \frac{\sqrt{\pi} e^2 K}{8 m_e c} \nu_L^{\frac{p+2}{2}} \nu^{-\frac{p+4}{2}} f_\kappa(p) \tag{3.12}$$

where the function $f_\kappa(p)$ is:

$$
\begin{aligned}
f_\kappa(p) &= 3^{\frac{p+1}{2}} \frac{\Gamma\left(\frac{3p+22}{12}\right) \Gamma\left(\frac{3p+2}{12}\right) \Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+8}{4}\right)} \\
&\sim 3^{\frac{p+1}{2}} \left(\frac{1.8}{p^{0.7}} + \frac{p^2}{40}\right)
\end{aligned}
\tag{3.13}
$$

The simple fitting function is accurate at the per cent level.

3.1.3 Specific intensity

Simple radiative tranfer allows to calculate the specific intensity:

$$I(\nu) = \frac{\epsilon_s(\nu)}{\kappa_\nu} \left(1 - e^{-\tau_\nu}\right) \tag{3.14}$$

where the absorption optical depth $\tau_\nu \equiv \kappa_\nu R$ and R is the size of the emitting region. When $\tau_\nu \gg 1$, the esponential term vanishes, and the intensity is simply the ratio between the emissivity and the absorption coefficient. This is the self–absorbed, ot thick, regime. In this case, since both $\epsilon_s(\nu)$ and κ_ν depends linearly upon K, the resulting self–absorbed intensity does not depend on the normalization of the particle density K:

$$I(\nu) = \frac{2 m_e}{\sqrt{3}\, \nu_L^{1/2}} f_I(p) \left(1 - e^{-\tau_\nu}\right) \tag{3.15}$$

we can thus see that the slope of the self–absorbed intensity does not depend on p. Its normalization, however, does (albeit weakly) depend on p through the function $f_I(p)$, which in the case of averaging over an isotropic pitch angle distribution is given by:

$$
\begin{aligned}
f_I(p) = \frac{1}{p+1} &= \frac{\Gamma\left(\frac{3p-1}{12}\right) \Gamma\left(\frac{3p+19}{12}\right) \Gamma\left(\frac{p+5}{4}\right) \Gamma\left(\frac{p+8}{4}\right)}{\Gamma\left(\frac{3p+22}{12}\right) \Gamma\left(\frac{3p+2}{12}\right) \Gamma\left(\frac{p+7}{4}\right) \Gamma\left(\frac{p+6}{4}\right)} \\
&\sim \frac{5}{4\, p^{4/3}}
\end{aligned}
\tag{3.16}
$$

where again the simple fitting function is accurate at the level of 1 per cent.

3.1.4   Self–absorption frequency

The self–absorption frequency ν_t is defined by $\tau_{\nu_t} = 1$:

$$\nu_t = \nu_L \left[\frac{\sqrt{\pi} e^2 RK}{8 m_e c \nu_L} f_\kappa(p) \right]^{\frac{4}{p+4}} = \nu_L \left[\frac{\pi \sqrt{\pi}}{4} \frac{eRK}{B} f_\kappa(p) \right]^{\frac{2}{p+4}} \qquad (3.17)$$

Note that the term in parenthesis is adimensional, and since RK has units of the inverse of a surface, then e/B has the dimension of a surface. In fact we have already discussed that this is the synchrotron absorption cross section of a relativistic electron of energy $\gamma m_e c^2$ absorbing photons at the fundamental frequency ν_L / γ.

The random Lorentz factor γ_t of the electrons absorbing (and emitting) photons with frequency ν_t is $\gamma_t \sim [3\nu_t/(4\nu_L)]^{1/2}$.

3.1.5   Synchrotron peak

In a $F(\nu)$ plot, the synchrotron spectrum peaks close to ν_t, at a frequency $\nu_{s,p}$ given by solving

$$\frac{dI(\nu)}{d\nu} = 0 \rightarrow \frac{d}{d\nu} \left[\nu^{5/2} \left(1 - e^{-\tau_\nu} \right) \right] = 0 \qquad (3.18)$$

which is equivalent to solve the equation:

$$\exp\left(\tau_{\nu_{s,p}} \right) - \frac{p+4}{5} \tau_{\nu_{s,p}} - 1 = 0 \qquad (3.19)$$

whose solution can be approximated by

$$\tau_{\nu_{s,p}} \sim \frac{2}{5} p^{1/3} \ln p \qquad (3.20)$$

Chapter 4

Compton scattering

4.1 Introduction

The simplest interaction between photons and free electrons is *scattering*.
When the energy of the incoming photons (as seen in the comoving frame
of the electron) is small with respect to the electron rest mass–energy, the
process is called *Thomson scattering*, which can be described in terms of
classical electro–dynamics. As the energy of the incoming photons increases
and becomes comparable or greater than $m_e c^2$, a quantum treatment is
necessary (Klein–Nishina regime).

4.2 The Thomson cross section

Assume an electron at rest, and an electromagnetic wave of frequency $\nu \ll$
$m_e c^2/h$. Assume also that the incoming wave is completely linearly po-
larized. In order to neglect the magnetic force $(e/c)(\mathbf{v} \times B)$ we must also
require that the oscillation velocity $v \ll c$. This in turn implies that the
incoming wave has a sufficiently low amplitude. The electron start to os-
cillate in response to the varying electric force eE, and the average square
acceleration during one cicle of duration $T = 1/\nu$ is

$$\langle a^2 \rangle \;=\; \frac{1}{T} \int_0^T \frac{e^2 E_0^2}{m_e^2} \sin^2(2\pi \nu t) \, dt \;=\; \frac{e^2 E_0^2}{2m_e^2} \tag{4.1}$$

The emitted power per unit solid angle is given by the Larmor formula
$dP/d\Omega = e^2 a^2 \sin^2 \Theta/(4\pi c^3)$ where Θ is the angle between the acceleration
vector and the propagation vector of the emitted radiation. Please note
that Θ is *not* the scattering angle, which is instead the angle between the
incoming and the scattered wave (or photon). We then have

$$\frac{dP_e}{d\Omega} \;=\; \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \, \sin^2 \Theta \tag{4.2}$$

The scattered radiation is completely linearly polarized in the plane defined by the incident polarization vector and the scattering direction. The flux of the incoming wave is $S_i = cE_0^2/(8\pi)$. The differential cross section of the process is then

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} = \frac{dP_e/d\Omega}{S_i} = r_0^2 \sin^2 \Theta \tag{4.3}$$

where $r_0 \equiv e^2/(m_e c^2)$ is the *classic electron radius*, $r_0 = 2.82 \times 10^{-13}$ cm. We see that the scattered pattern of a completely polarized incoming wave is a torus, with axis along the acceleration direction. The total cross section can be derived in a similar way, but considering the Larmor formula integrated over the solid angle $[P = 2e^2a^2/(3c^3)]$. In this way the total cross section is

$$\sigma_{\text{pol}} = \frac{P_e}{S_i} = \frac{8\pi}{3}r_0^2 \tag{4.4}$$

Note that the classical electron radius can also be derived by equating the energy of the associated electric field to the electron rest mass–energy:

$$m_e c^2 = \int_{a_o}^{\infty} \frac{E^2}{8\pi} 4\pi r^2 dr = \int_{a_o}^{\infty} \frac{e^2}{2r^2} dr \rightarrow a_0 = \frac{1}{2}\frac{e^2}{m_e c^2} \tag{4.5}$$

Why is a_0 slightly different from r_0? Because there is an intrinsic uncertainty related to the distribution of the charge within (or throughout the surface of) the electron. See the discussion in Vol. 2, chapter 28.3 of "The Feynman Lectures on Physics", about the fascinating idea that the mass of the electron is all electromagnetic.

4.2.1 Why the peanut shape?

The scattering of a completely unpolarized incoming wave can be derived by assuming that the incoming radiation is the sum of two orthogonal completely linearly polarized waves, and then summing the associated scattering patterns. Since we have the freedom to chose the orientations of the two polarization planes, it is convenient to chose one of these planes as the one defined by the incident and scattered directions, and the other one perpendicular to this plane. The scattering can be then regarded as the sum of two independent scattering processes, one with emission angle Θ, the other with $\pi/2$. If we note that the scattering angle (i.e. the angle between the scattered wave and the incident wave) is $\theta = \pi/2 - \Theta$, we have

$$\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} &= \frac{1}{2}\left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}}\right] \\
&= \frac{1}{2}r_0^2(1 + \sin^2 \Theta) \\
&= \frac{1}{2}r_0^2(1 + \cos^2 \theta) \tag{4.6}
\end{aligned}$$
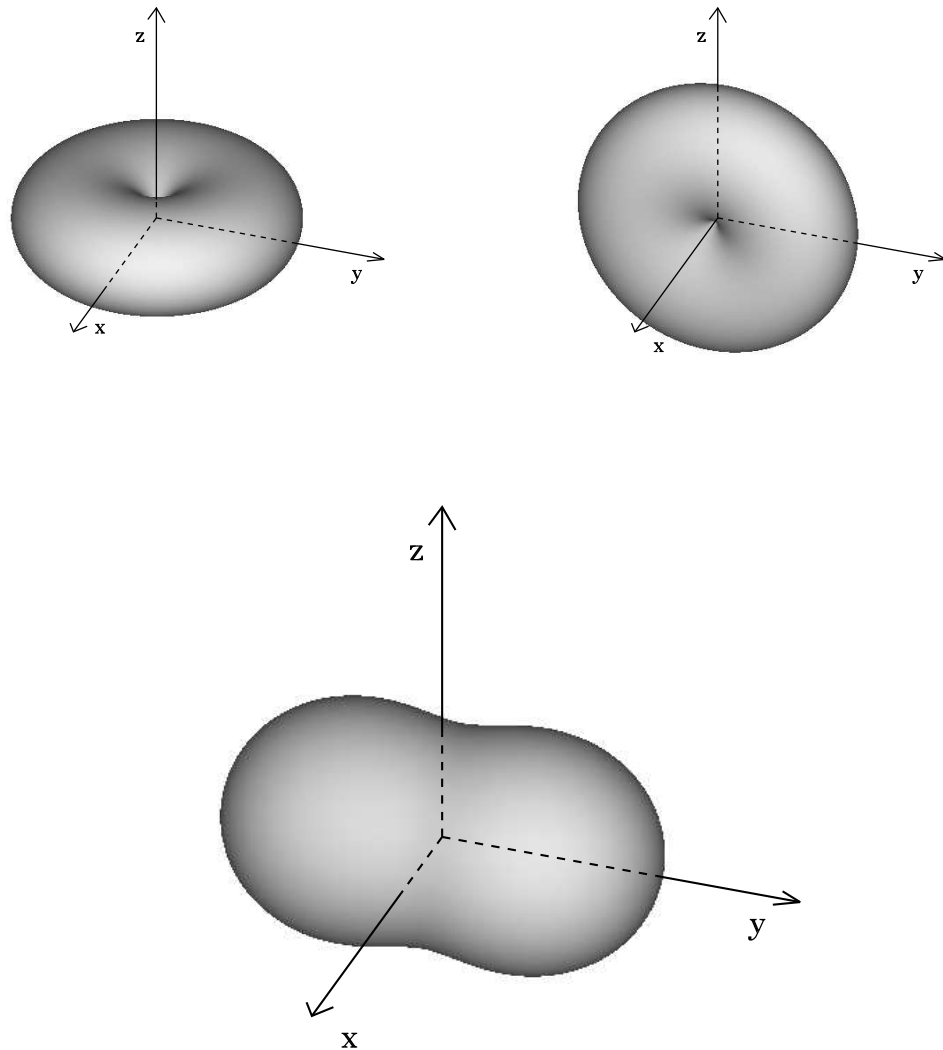
Figure 4.1: Photons are coming along the y–axis. The top panels shows the pattern of the scattered radiation for photons completely linearly polarized along the z–axis (left) and along the x–axis (right). The sum of the two torii corresponds to the pattern for unpolarized radiation (bottom panel). This explains why we have a "peanut" shape, elongated along the velocity vector of the incoming photons. Courtesy of Davide Lazzati.

In this case we see that the cross section depends only on the scattering angle θ. The pattern of the scattered radiation is then the superposition of two orthogonal "tori" (one for each polarization direction), as illustrated in Fig. 4.1. When scattering completely linearly polarized radiation, only one "torus" survives. Instead, when scattering unpolarized radiation, some polarization is introduced, because of the difference between the two "tori" patterns. Both terms of the RHS of Eq. 4.6 refer to completely polarized scattered waves (but in two perpendicular planes). The difference between these two terms is then associated to the introduced polarization, which is then

$$\Pi = \frac{1 - \cos^2 \theta}{1 + \cos^2 \theta} \tag{4.7}$$

The above discussion help to understand why the scattering process introduces some polarization, which is maximum (100%) if the angle between the incoming and the scattered photons is 90° (only one torus contributes), and zero for 0° and 180°, where the two torii give the same contribution.

The total cross section, integrated over the solid angle, is the same as that for polarized incident radiation (Eq. 4.4) since the electron at rest has no preferred defined direction. This is the Thomson cross section:

$$\begin{aligned} \sigma_{\mathrm{T}} &= \int \left(\frac{d\sigma}{d\Omega} \right)_{\mathrm{unpol}} d\Omega = \frac{2\pi r_0^2}{2} \int (1 + \cos^2 \theta) d\cos\theta = \frac{8\pi}{3} r_0^2 \\ &= 6.65 \times 10^{-25} \ \mathrm{cm}^2 \end{aligned} \tag{4.8}$$

4.3   Direct Compton scattering

In the previous section we considered the scattering process as an interaction between an electron and an *electromagnetic wave*. This required $h\nu \ll m_e c^2$. In the general case the quantum nature of the radiation must be taken into account. We consider then the scattering process as a collision between the electron and the *photon*, and apply the conservation of energy and momentum to derive the energy of the scattered photon. It is convenient to measure energies in units of $m_e c^2$ and momenta in units of $m_e c$.

Consider an electron at rest and an incoming photon of energy x_0, which becomes x_1 after scattering. Let θ be the angle between the incoming and outgoing photon directions. This defines the scattering plane. Momentum conservation dictates that also the momentum vector of the electron, after the scattering, lies in the same plane. Conservation of energy and conservation of momentum along the x and y axis gives:

$$x_1 = \frac{x_0}{1 + x_0(1 - \cos\theta)} \tag{4.9}$$

Note that, for $x_0 \gg 1$ and $\cos\theta \neq 1$, $x_1 \rightarrow (1 - \cos\theta)^{-1}$. In this case the scattered photon carries information about the scattering angle, rather than

about the initial energy. As an example, for $\theta = \pi$ and $x_0 \gg 1$, the final energy is $x_1 = 0.5$ (corresponding to 255 keV) independently of the exact value of the initial photon energy. Note that for $x_0 \ll 1$ the scattered energy $x_1 \simeq x_0$, as assumed in the classical Thomson scattering. The energy shift implied by Eq. 4.9 is due to the recoil of the electron originally at rest, and becomes significant only when x_0 becomes comparable with 1 (or more). When the energy of the incoming photon is comparable to the electron rest mass, another quantum effect appears, namely the energy dependence of the cross section.

4.4 The Klein–Nishina cross section

The Thomson cross section is the classical limit of the more general Klein–Nishina cross section (here we use x as the initial photon energy, instead of x_0, for simplicity):

$$\frac{d\sigma_{\mathrm{KN}}}{d\Omega} = \frac{3}{16\pi} \sigma_{\mathrm{T}} \left(\frac{x_1}{x}\right)^2 \left(\frac{x}{x_1} + \frac{x_1}{x} - \sin^2\theta\right) \qquad (4.10)$$

This is a compact form, but there appears dependent quantities, as $\sin\theta$ is related to x and x_1. By inserting Eq. 4.9, we arrive to

$$\frac{d\sigma_{\mathrm{KN}}}{d\Omega} = \frac{3}{16\pi} \frac{\sigma_{\mathrm{T}}}{[1 + x(1 - \cos\theta)]^2} \left[x(1 - \cos\theta) + \frac{1}{1 + x(1 - \cos\theta)} + \cos^2\theta\right]$$
$$(4.11)$$

In this form, only independent quantities appear (i.e. there is no x_1). Note that the cross section becomes smaller for increasing x and that it coincides with $d\sigma_{\mathrm{T}}/d\Omega$ for $\theta = 0$ (for this angle $x_1 = x$ independently of x). This however corresponds to a vanishingly small number of interactions, since $d\Omega \to 0$ for $\theta \to 0$).

Integrating Eq. 4.11 over the solid angle, we obtain the total Klein–Nishina cross section:

$$\sigma_{\mathrm{KN}} = \frac{3}{4} \sigma_{\mathrm{T}} \left\{\frac{1 + x}{x^3} \left[\frac{2x(1 + x)}{1 + 2x} - \ln(1 + 2x)\right] + \frac{1}{2x} \ln(1 + 2x) - \frac{1 + 3x}{(1 + 2x)^2}\right\}$$
$$(4.12)$$

Asimptotic limits are:

$$\sigma_{\mathrm{KN}} \simeq \sigma_{\mathrm{T}} \left(1 - 2x + \frac{26x^2}{5} + ...\right); \quad x \ll 1$$

$$\sigma_{\mathrm{KN}} \simeq \frac{3}{8} \frac{\sigma_{\mathrm{T}}}{x} \left[\ln(2x) + \frac{1}{2}\right]; \qquad x \gg 1 \qquad (4.13)$$

The direct Compton process implies a transfer of energy from the photons to the electrons. It can then be thought as an heating mechanism. In the next
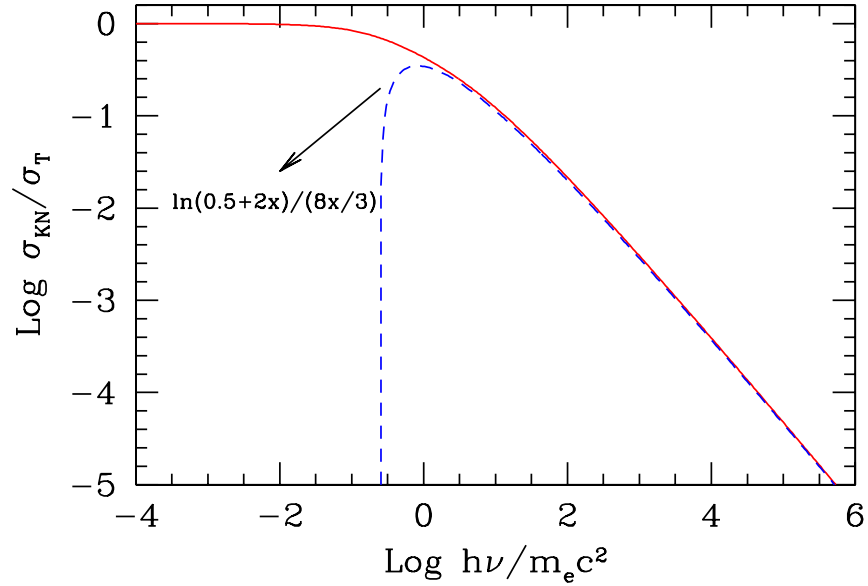
Figure 4.2: The total Klein–Nishina cross section as a function of energy. The dashed line is the approximation at high energies as given in Eq. 4.13.
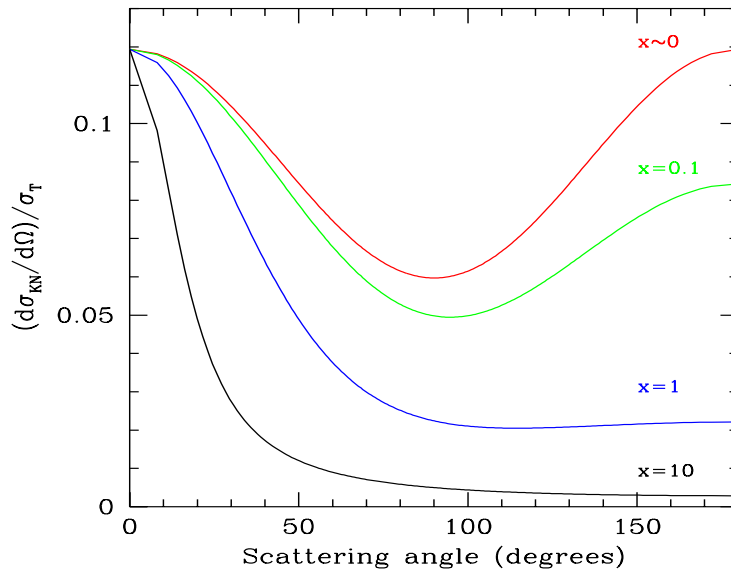

Figure 4.3: The differential Klein–Nishina cross section (in units of σ_T), for different incoming photon energies. Note how the scattering becomes preferentially forward as the energy of the photon increases.
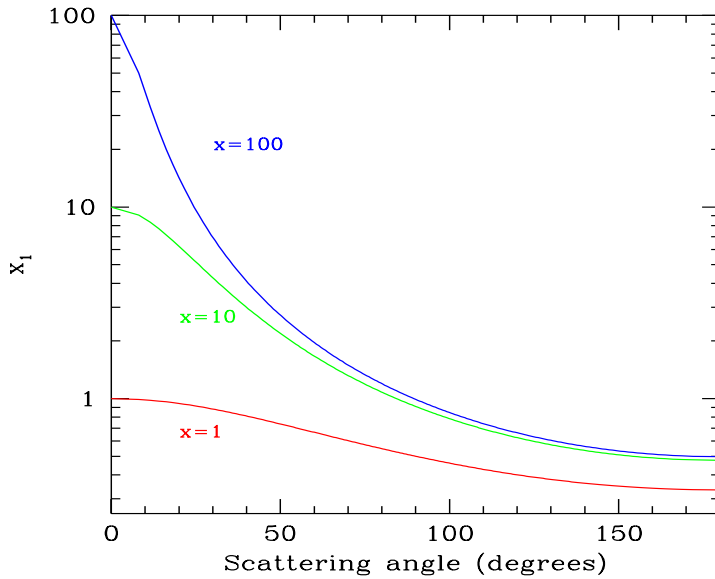
Figure 4.4: Scattered photons energies as a function of the scattering angle, for different incoming photon energies. Note that, for $x \gg 1$ and for large scattering angle, the scattered photon energies becomes $x_1 \sim 1/2$, independent of the initial photon energy x.

subsection we discuss the opposite process, called *inverse Compton scattering*, in which hot electrons can transfer energy to low frequency photons.

We have so far neglected the momentum exchange between radiation and the electron. One can see, even classically, that there must be a net force acting along the direction of the wave if one considers the action of the magnetic field of the wave. In fact the Lorentz force $e\mathbf{v} \times \mathbf{B}$ is *always* directed along the direction of the wave (here $\mathbf{v}$ is the velocity along the $\mathbf{E}$ field). This explains the fact that light can exert a pressure, even classically.

4.4.1 Another limit

We have mentioned that, in order for the magnetic Lorentz force to be negligible, the electron must have a transverse (perpendicular to the incoming wave direction) velocity $\ll c$. Considering a wave of frequency ω and electric field $E = E_0 \sin(\omega t)$, this implies that:

$$\frac{v_\perp}{c} = \int_0^{T/2} \frac{eE_0}{cm_e} \sin(\omega t)dt = \frac{2eE_0}{m_e c \omega} \ll 1 \qquad (4.14)$$

This means that the scattering process can be described by the Thomson cross section if the wave have a sufficiently low amplitude and a not too small frequency (i.e. for very small frequencies the electric field of the wave accelerates the electron for a long time, and then to large velocities).

4.4.2 Pause

Now pause, and ask if there are some ways to apply what we have done up to now to real astrophysical objects.

- The Eddington luminosity is derived with the Thomson cross section, with the thought that it describes the smallest probability of interaction between matter and radiation. But the Klein–Nishina cross section can be even smaller, as long as the source of radiation emits at high energy. What are the consequences? If you have forgotten the definition of the Eddington luminosity, here it is:

$$L_{\mathrm{Edd}} = \frac{4\pi GM m_{\mathrm{p}}}{\sigma_{\mathrm{T}}} = 1.5 \times 10^{38} \frac{M}{M_{\odot}} \ \mathrm{erg \ s^{-1}} \qquad (4.15)$$

- In Nova Muscae, some years ago a (transient) annihilation line was detected, together with another feature (line–like) at 200 keV. What can this feature be?

- It seems that high energy radiation can suffer less scattering and therefore can propagate more freely through the universe. Is that true? Can you think to other processes that can kill high energy photons in space?

- Suppose to have an astrophysical source of radiation very powerful above say – 100 MeV. Assume that at some distance there is a very efficient "reflector" (i.e. free electrons) and that you can see the scattered radiation. Can you guess the spectrum you receive? Does it contain some sort of "pile–up" or not? Will this depend upon the scattering angle?

4.5   Inverse Compton scattering

When the electron is not at rest, but has an energy greater that the typical photon energy, there can be a transfer of energy from the electron to the photon. This process is called *inverse* Compton to distinguish it from the *direct* Compton scattering, in which the electron is at rest, and it is the photon to give part of its energy to the electron.

We have two regimes, that are called the *Thomson* and the *Klein–Nishina* regimes. The difference between them is the following: we go in the frame where the electron is at rest, and in that frame we calculate the energy of the incoming photon. If the latter is smaller than $m_e c^2$ we are in the Thomson regime. In this case the recoil of the electron, even if it always exists, is small, and can be neglected. In the opposite case (photon energies larger than $m_e c^2$), we are in the Klein–Nishina one, and we cannot neglect the recoil. As we shall see, in both regimes the typical photon *gain* energy, even

if there will always be some arrangements of angles for which the scattered photon looses part of its energy.

4.5.1 Thomson regime

Perhaps, a better name should be "inverse Thomson" scattering, as will appear clear shortly.

4.5.2 Typical frequencies

In the frame K' comoving with the electron, the incoming photon energy is

$$x' = x\gamma(1 - \beta\cos\psi) \tag{4.16}$$

where ψ is the angle between the electron velocity and the photon direction (see Fig 4.5). Then if $x' \ll 1$, we are in the Thomson regime. In the rest frame of the electron the scattered photon will have the same energy x_1' as before the scattering, independent of the scattering angle. Then

$$x_1' = x' \tag{4.17}$$

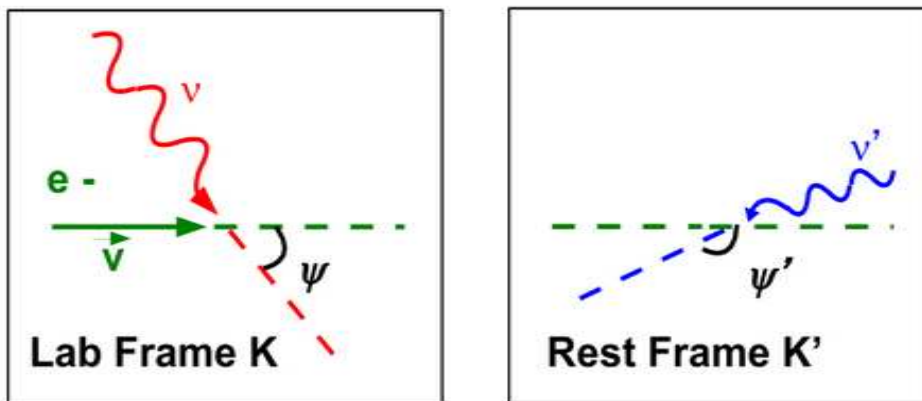This photon will be scattered at an angle ψ_1' with respect to the electron

Figure 4.5: In the lab frame an electron is moving with velocity **v**. Its velocity makes an angle ψ with an incoming photon of frequency ν. In the frame where the electron is at rest, the photon is coming from the front, with frequency ν', making an angle ψ' with the direction of the velocity.

velocity. Going back to K the observer sees

$$x_1 = x_1' \gamma(1 + \beta \cos \psi_1') \tag{4.18}$$

Recalling Eq. 1.15, for the transformation of angles:

$$\cos \psi_1' = \frac{\beta + \cos \psi_1}{1 + \beta \cos \psi_1} \tag{4.19}$$

we arrive to the final formula:

$$x_1 = x \frac{1 - \beta \cos \psi}{1 - \beta \cos \psi_1} \tag{4.20}$$

Now all quantities are calculated in the lab–frame.
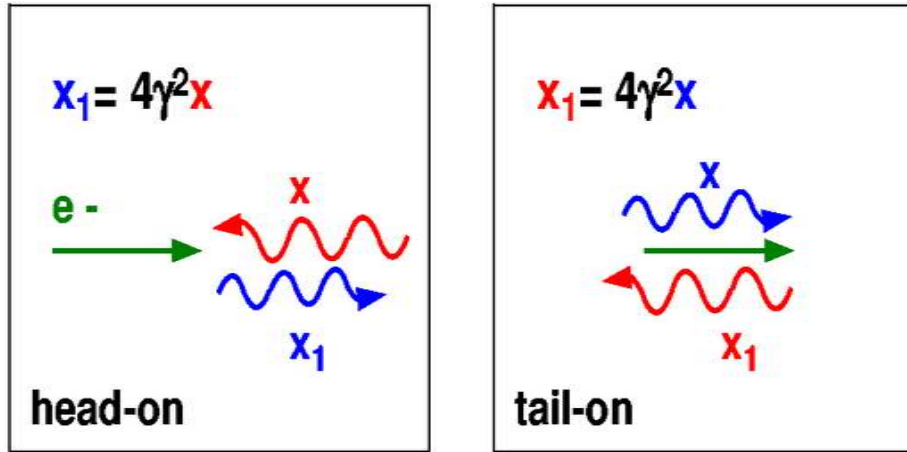
Figure 4.6: Minimum and maximum scattered frequencies. The maximum occurs for head-on collisions, the minimum for tail–on ones. These two frequencies are one the inverse of the other.

Let us see the minimum and maximum energies. The maximum is when $\psi = \pi$ (head on collision), and when $\psi_1 = 0$ (the photon is scattered along the electron velocity vector). In these are head–on collisions:

$$x_1 = x \frac{1 + \beta}{1 - \beta} = \gamma^2(1 + \beta)^2 x \rightarrow 4\gamma^2 x; \quad \text{head} - \text{on} \tag{4.21}$$

where the last step is valid if $\gamma \gg 1$. The other extreme is for $\psi_1 = \pi$ and $\psi = 0$. In this case the incoming photon "comes from behind" and bounces back. In these "tail–on" collisions:

$$x_1 = x \frac{1 - \beta}{1 + \beta} = \frac{x}{\gamma^2(1 + \beta)^2} \rightarrow \frac{x}{4\gamma^2}; \quad \text{tail} - \text{on} \tag{4.22}$$

where again the last step is valid if $\gamma \gg 1$. Another typical angle is $\sin \psi_1 = 1/\gamma$, corresponding to $\cos \psi_1 = \beta$. This corresponds to the aperture angle of the beaming cone. For this angle:

$$x_1 = \frac{1 - \beta \cos \psi}{1 - \beta^2} x = \gamma^2 (1 - \beta \cos \psi) x; \quad \text{beaming cone} \qquad (4.23)$$

which becomes $x_1 = x/(1 + \beta)$ for $\psi = 0$, $x_1 = \gamma^2 x$ for $\psi = \pi/2$ and $x_1 = \gamma^2(1 + \beta)x$ for $\psi = \pi$.

For an isotropic distribution of incident photons and for $\gamma \gg 1$ the average photon energy after scattering is (see Eq. 4.45):

$$\langle x_1 \rangle = \frac{4}{3} \gamma^2 x \qquad (4.24)$$

Total loss rate

We can simply calculate the rate of scatterings per electron considering all quantities in the lab–frame. Let $n(\epsilon)$ be the density of photons of energy $\epsilon = h\nu$, v the electron velocity and ψ the angle between the electron velocity and the incoming photon. For mono–directional photon distributions, we have:

$$\frac{dN}{dt} = \int \sigma_{\mathrm{T}} v_{\mathrm{rel}} n(\epsilon) d\epsilon \qquad (4.25)$$

$v_{\mathrm{rel}} = c - v \cos \psi$ is the relative velocity between the electron and the incoming photons. We then have

$$\frac{dN}{dt} = \int \sigma_{\mathrm{T}} c (1 - \beta \cos \psi) n(\epsilon) d\epsilon \qquad (4.26)$$

Note that the rate of scatterings in the lab frame, when the electron and/or photon are anisotropically distributed, can be described by an effective cross section $\sigma_{\mathrm{eff}} \equiv \int \sigma_{\mathrm{T}} (1 - \beta \cos \psi) d\Omega/4\pi$. For photons and electrons moving in the same direction the scattering rate (hence, the effective optical depth) can be greatly reduced.

The power contained in the scattered radiation is then

$$\frac{dE_\gamma}{dt} = \frac{\epsilon_1 dN}{dt} = \sigma_{\mathrm{T}} c \int \frac{(1 - \beta \cos \psi)^2}{1 - \beta \cos \psi_1} \epsilon n(\epsilon) d\epsilon \qquad (4.27)$$

Independently of the incoming photon angular distribution, the average value of $1 - \beta \cos \psi_1$ can be calculated recalling that, in the rest frame of the electron, the scattering has a backward–forward symmetry, and therefore $\langle \cos \psi_1' \rangle = \pi/2$. The average value of $\cos \psi_1$ is then β, leading to $\langle 1 - \beta \cos \psi_1 \rangle = 1/\gamma^2$. We therefore obtain

$$\frac{dE_\gamma}{dt} = \sigma_{\mathrm{T}} c \gamma^2 \int (1 - \beta \cos \psi)^2 \epsilon n(\epsilon) d\epsilon \qquad (4.28)$$

If the incoming photons are isotropically distributed, we can average out $(1 - \beta \cos \psi)^2$ over the solid angle, obtaining $1 + \beta^2/3$. The power produced is then

$$\frac{dE_\gamma}{dt} = \sigma_{\mathrm{T}} c \gamma^2 \left(1 + \frac{\beta^2}{3} \right) U_{\mathrm{r}} \tag{4.29}$$

where

$$U_{\mathrm{r}} = \int \epsilon n(\epsilon) d\epsilon \tag{4.30}$$

is the energy density of the radiation before scattering. This is the power contained in the scattered radiation. To calculate the energy loss rate of the electron, we have to subtract the initial power of the radiation eventually scattered

$$P_{\mathrm{c}}(\gamma) \equiv \frac{dE_e}{dt} = \frac{dE_\gamma}{dt} - \sigma_{\mathrm{T}} c U_{\mathrm{r}} = \frac{4}{3} \sigma_{\mathrm{T}} c \gamma^2 \beta^2 U_{\mathrm{r}} \tag{4.31}$$

A simple way to remember Eq. 4.31 is:

$$
\begin{aligned}
P_{\mathrm{c}}(\gamma) &= \left(\frac{\text{\# of collisions}}{\sec} \right) \text{(average phot. energy after scatt.)} \\
&= \left(\sigma_{\mathrm{T}} c \frac{U_{\mathrm{r}}}{\langle h\nu \rangle} \right) \left(\frac{4}{3} \langle h\nu \rangle \gamma^2 \right)
\end{aligned}
\tag{4.32}
$$

Note the similarity with the synchrotron energy loss. The two energy loss rates are identical, once the radiation energy density is replaced by the magnetic energy density U_{B}. Therefore, if relativistic electrons are in a region with some radiation and magnetic energy densities, they will emit by both the synchrotron and the Inverse Compton scattering processes. The ratio of the two luminosities will be

$$\frac{L_{\mathrm{syn}}}{L_{\mathrm{IC}}} = \frac{P_{\mathrm{syn}}}{P_{\mathrm{c}}} = \frac{U_{\mathrm{B}}}{U_{\mathrm{r}}} \tag{4.33}$$

where we have set $dE_{\mathrm{IC}}/dt = dE_e/dt$. This is true unless one of the two processes is inhibited for some reason. For instance:

- At (relatively) low energies, electrons could emit *and absorb* synchrotron radiation, so the synchrotron cooling is compensated by the heating due to the absorption process.

- At high energies, electrons could scatter in the Klein–Nishina regime: in this case, since the cross section is smaller, they will do less scatterings, and cool less.

But let us go back to Eq. 4.28, that is the starting point when dealing with *anisotropic* seed photon distributions. Think for instance to an accretion disk as the producer of the seed photons for scattering, and some cloud of relativistic electrons above the disk. If the cloud is not that distant, and
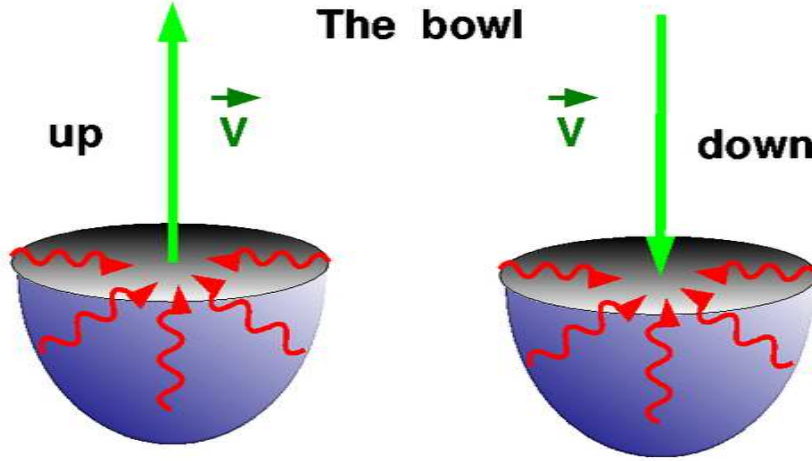
Figure 4.7: In the center of a semi–sphere (the "bowl") we have relativistic electrons going down and going up, all with the same γ. Since the seed photon distribution is anisotropic, so is the scattered radiation and power. The losses of the electron going down are 7 times larger than those of the electron going up (if $\gamma \gg 1$). Since almost all the radiation is produced along the velocity vector of the electrons, also the downward radiation is 7 times more powerful than the upward radiation.

it is small with respect to the disk size, then this case is completely equal to the case of having a little cloud of relativistic electrons located at the center of a semi–sphere. That is, we have the "bowl" case illustrated in Fig 4.7. Just for fun, let us calculate the total power emitted by an electron going "up" and by its brother (i.e. it has the same γ) going down. Using Eq. 4.28 we have:

$$\frac{P_{\text{down}}}{P_{\text{up}}} = \frac{\int_{-1}^{0}(1-\beta\mu)^2 d\mu}{\int_{0}^{-1}(1-\beta\mu)^2 d\mu} = \frac{1+\beta+\beta^2/3}{1-\beta+\beta^2/3} \rightarrow 7 \qquad (4.34)$$

where $\mu \equiv \cos\psi$ and the last step assumes $\beta \rightarrow 1$. Since almost all the radiation is produced along the velocity vector of the electrons, also the downward radiation is more powerful than the upward radiation (i.e. 7 times more powerful for $\gamma \gg 1$). What happens if the cloud of electrons is located at some height above the bowl? Will the $P_{\text{down}}/P_{\text{up}}$ be more or less?

4.5.3   Cooling time and compactness

The cooling time due to the inverse Compton process is

$$t_{\rm IC} \; = \; \frac{E}{dE_e/dt} \; = \; \frac{3\gamma m_e c^2}{4\sigma_{\rm T} c\gamma^2\beta^2 U_{\rm r}} \; \sim \; \frac{3m_e c^2}{4\sigma_{\rm T} c\gamma U_{\rm r}}; \qquad \gamma\epsilon \ll m_{\rm e}c^2 \qquad (4.35)$$

This equation offers the opportunity to introduce an important quantity, namely *the compactness* of an astrophysical source, that is essentially the luminosity L over the size R ratio. Consider in fact how $U_{\rm r}$ and L are related:

$$U_{\rm r} \; = \; \frac{L}{4\pi R^2 c} \qquad (4.36)$$

Although this relation is almost universally used, there are subtleties. It is surely valid if we measured $U_{\rm r}$ *outside* the source, at a distance R from its center. In this case $4\pi R^2 c$ is simply the volume of the shell crossed by the source radiation in one second. But if we are *inside* an homogeneous, spherical transparent source, a better way to calculate $U_{\rm r}$ is to think to the average time needed to the typical photon to exit the source. This is $t_{\rm esc} = 3R/(4c)$. It is less than R/c because the typical photon is not born at the center (there is more volume close to the surface). If $V = (4\pi/3)R^3$ is the volume, we can write:

$$U_{\rm r} \; = \; \frac{L}{V}\, t_{\rm esc} \; = \; \frac{3L}{4\pi R^3}\frac{3R}{4c} \; = \; \frac{9L}{16\pi R^2 c} \qquad (4.37)$$

This is greater than Eq. 4.36 by a factor 9/4. Anyway, let us be conventional and insert Eq. 4.36 in Eq. 4.35:

$$t_{\rm IC} \; = \; \frac{3\pi m_e c^2 R^2}{\sigma_{\rm T}\gamma L} \; \to \; \frac{t_{\rm IC}}{R/c} \; = \; \frac{3\pi}{\gamma}\frac{m_e c^3 R}{\sigma_{\rm T} L} \; \equiv \; \frac{3\pi}{\gamma}\frac{1}{\ell} \qquad (4.38)$$

where the dimensionless compactness ℓ is defined as

$$\ell \; = \; \frac{\sigma_{\rm T} L}{m_{\rm e} c^3 R} \qquad (4.39)$$

For ℓ close or larger than unity, we have that even low energy electrons cool by the Inverse Compton process in less than a light crossing time R/c.

There is another reason why ℓ is important, related to the fact that it directly measures the optical depth (hence the probability to occur) of the photon–photon collisions that lead to the creation of electron–positron pairs. The compactness is one of the most important physical parameters when studying compact high energy compact sources (X–ray binaries, AGNs and Gamma Ray Bursts).

4.5.4 Single particle spectrum

As we did for the synchrotron spectrum, we will not repeat the exact derivation of the single particle spectrum, but we try to explain why the typical frequency of the scattered photon is a factor γ^2 larger than the frequency of the incoming photon. Here are the steps to consider:

1. Assume that the relativistic electron travels in a region where there is a radiation energy density U_r made by photons which we will take, for simplicity, monochromatic, therefore all having a dimensionless frequency $x = h\nu/m_e c^2$.

2. In the frame where the electron is at rest, half of the photon appear to come from the front, within an angle $1/\gamma$.

3. The typical frequency of these photon is $x' \sim \gamma x$ (it is twice that for photons coming exactly head on).

4. Assuming that we are in the Thomson regime means that i) $x' < 1$; ii) the cross section is the Thomson one; iii) the frequency of the scattered photon is the same of the incoming one, i.e. $x_1' = x' \sim \gamma x$, and iv) the pattern of the scattered photons follows the angular dependence of the cross section, therefore the "peanut".

5. Independently of the initial photon direction, and therefore independently of the frequencies seen by the electrons, all photons after scatterings are isotropized. This means that all observer (at any angle ψ_1' in this frame see the same spectrum, and the same typical frequency. Half of the photons are in the semi-sphere with $\psi_1' \leq \pi/2$.

6. Now we go back to the lab–frame. Those photons that had $\psi_1' \leq \pi/2$ now have $\psi_1 \leq 1/\gamma$. Their typical frequency if another factor γ greater than what they had in the rest frame, therefore

$$x_1 \sim \gamma^2 x \tag{4.40}$$

This is the typical Inverse Compton frequency.

The exact derivation can be found e.g. in Rybicki & Lightman (1979) and in Blumenthal & Gould (1970). We report here the final result, valid for a monochromatic and isotropic seed photons distribution, characterized by a specific intensity

$$\frac{I(x)}{x} = \frac{I_0}{x}\delta(x - x_0) \tag{4.41}$$

Note that $I(x)/x$ is the analog of the normal intensity, but it is associated with the number of photons. If we have n electrons per cubic centimeter we have:

$$\epsilon_{IC}(x_1) = \frac{\sigma_T n I_0 (1 + \beta)}{4\gamma^2 \beta^2 x_0} F_{IC}(x_1) \tag{4.42}$$

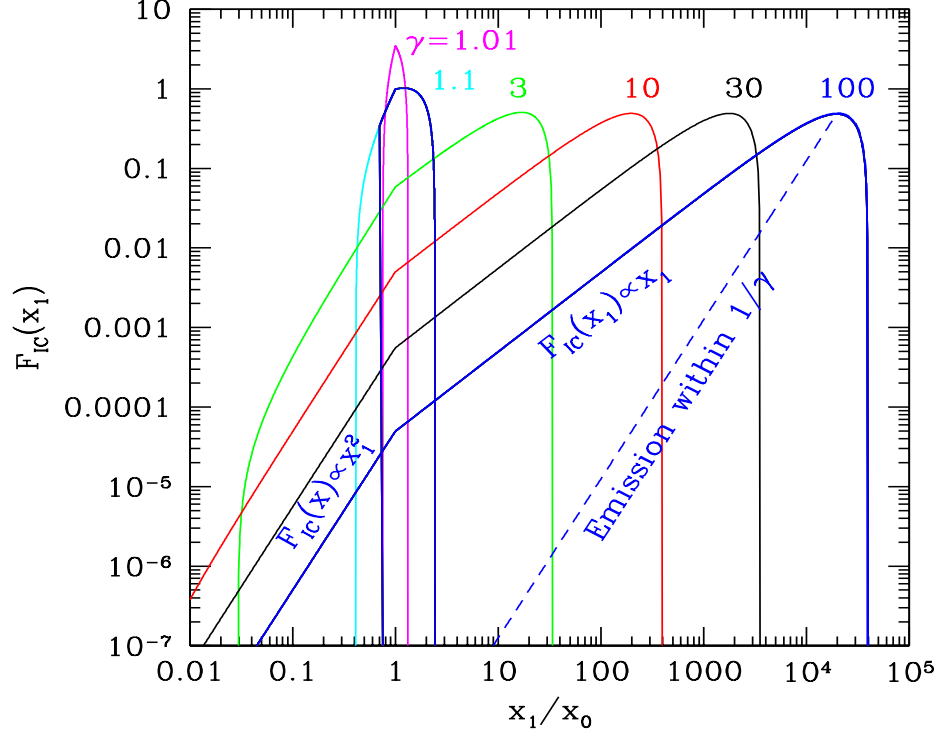The function $F_{\rm IC}$ contains all the frequency dependence:

Figure 4.8: Spectrum emitted by the Inverse Compton process by electrons of different γ (as labeled) scattering an isotropic monochromatic radiation field of dimensionless frequency x_0. The dashed line corresponds to the spectrum emitted within the $1/\gamma$ beming cone: it always contains the 75% of the total power, for any γ. For $x_1 < x_0$ we have *downscattering*, i.e. the photons loose energy in the process. Note also the power law segments arising when $\gamma \gg 1$: $F_{\rm IC}(x_1) \propto x_1^2$ for the downscattering tail, and $F_{IC}(x_1) \propto x_1$ for the upscattering segment.

$$
\begin{aligned}
F_{\rm IC}(x_1) &= \frac{x_1}{x_0} \left[\frac{x_1}{x_0} - \frac{1}{(1+\beta)^2 \gamma^2} \right]; \quad \frac{1}{(1+\beta)^2 \gamma^2} < \frac{x_1}{x_0} < 1 \\
F_{\rm IC}(x_1) &= \frac{x_1}{x_0} \left[1 - \frac{x_1}{x_0} \frac{1}{(1+\beta)^2 \gamma^2} \right]; \quad 1 < \frac{x_1}{x_0} < (1+\beta)^2 \gamma^2 \ (4.43)
\end{aligned}
$$

The first line correspond to *downscattering*: the scattered photon has *less* energy than the incoming one. Note that in this case $F_{\rm IC}(x_1) \propto x_1^2$. The second line corresponds to *upscattering*: in this case $F_{\rm IC}(x_1) \propto x_1$ except for frequencies close to the maximum ones. The function $F_{\rm IC}(x_1)$ is shown in Fig. 4.8 for different values of γ. The figure shows also the spectrum of

the photons contained in the beaming cone $1/\gamma$: the corresponding power is always 75% of the total.

The average frequency of $F_{IC}(x_1)$ is

$$\langle x_1 \rangle = 2\gamma^2 x_0; \quad \text{energy spectrum} \tag{4.44}$$

This is the average frequency *of the energy spectrum*. We sometimes want to know the average energy *of the photons*, i.e. we have to calculate the average frequency of *the photon spectrum* $F_{IC}(x_1)/x_1$. This is:

$$\langle x_1 \rangle = \frac{4}{3}\gamma^2 x_0; \quad \text{photon spectrum} \tag{4.45}$$

4.6 Emission from many electrons

We have seen that the emission spectrum from a single particle is peaked, and the typical frequency is boosted by a factor γ^2. This is equal to the synchrotron case. Therefore we can derive the Inverse Compton emissivity as we did for the synchrotron one. Again, assume a power–law energy distribution for the relativistic electrons:

$$N(\gamma) = K\gamma^{-p} = N(E)\frac{dE}{d\gamma}; \quad \gamma_{min} < \gamma < \gamma_{max} \tag{4.46}$$

and assume that it describes an isotropic distribution of electrons. For simplicity, let us assume that the seed photons are isotropic and monochromatic, with frequency ν_0 (we now pass to real frequencies, since we are getting closer to the real world..). Since there is a strong link between the scattered frequency ν_c and the electron energy that produced it, we can set:

$$\nu_c = \frac{4}{3}\gamma^2\nu_0 \rightarrow \gamma = \left(\frac{3\nu_c}{4\nu_0}\right)^{1/2} \rightarrow \left|\frac{d\gamma}{d\nu}\right| = \frac{\nu_c^{-1/2}}{2}\left(\frac{3}{4\nu_o}\right)^{1/2} \tag{4.47}$$

Now, repeating the argument we used for synchrotron emission, we can state that the power lost by the electron of energy $\gamma m_e c^2$ within $m_e c^2 d\gamma$ goes into the radiation of frequency ν within $d\nu$. Since we will derive an emissivity (i.e. erg cm^{-3} s^{-1} sterad^{-1} Hz^{-1}) we must remember the 4π term (if the emission is isotropic). We can set:

$$\epsilon_c(\nu_c)d\nu_c = \frac{1}{4\pi}m_e c^2 P_c(\gamma)N(\gamma)d\gamma \tag{4.48}$$

This leads to:

$$\epsilon_c(\nu_c) = \frac{1}{4\pi}\frac{(4/3)^\alpha}{2}\sigma_T cK\frac{U_r}{\nu_0}\left(\frac{\nu_c}{\nu_0}\right)^{-\alpha} \tag{4.49}$$

Again, a power law, as in the case of synchrotron emission by a power law energy distribution. Again the same link between α and p:

$$\alpha = \frac{p-1}{2} \tag{4.50}$$

Of course, this is not a coincidence: it is because both the Inverse Compton and the synchrotron single electron spectra are peaked at a typical frequency that is a factor γ^2 greater than the starting one.

Eq. 4.49 becomes a little more clear if

- we express $\epsilon_c(\nu_c)$ as a function of the photon energy $h\nu_c$. Therefore $\epsilon_c(h\nu_c) = \epsilon_c(\nu_c)/h$;

- we multiply and divide by the source radius R;

- we consider a proxy for the scattering optical depth of the relativistic electrons setting $\tau_c \equiv \sigma_T K R$.

Then we obtain:

$$\epsilon_c(h\nu_c) = \frac{1}{4\pi} \frac{(4/3)^\alpha}{2} \frac{\tau_c}{R/c} \frac{U_r}{h\nu_0} \left(\frac{\nu_c}{\nu_0}\right)^{-\alpha} \tag{4.51}$$

In this way: τ_c (for $\tau_c < 1$) is the fraction of the seed photons $U_r/h\nu_0$ undergoing scattering in a time R/c, and $\nu_c/\nu_0 \sim \gamma^2$ is the average gain in energy of the scattered photons.

4.6.1 Non monochromatic seed photons

It is time to consider the more realistic case in which the seed photons are not monochromatic, but are distributed in frequency. This means that we have to integrate Eq. 4.49 over the incoming photon frequencies. For clarity, let us drop the subscript 0 in ν_0. We have

$$\epsilon_c(\nu_c) = \frac{1}{4\pi} \frac{(4/3)^\alpha}{2} \frac{\tau_c}{R/c} \nu_c^{-\alpha} \int_{\nu_{min}}^{\nu_{max}} \frac{U_r(\nu)}{\nu} \nu^\alpha d\nu \tag{4.52}$$

where $U_r(\nu)$ [erg cm^{-3} Hz^{-1}] is the specific radiation energy density at the frequency ν. The only difficulty of this integral is to find the correct limit of the integration, that, in general, *depend on ν_c*. Note also another interesting thing. We have just derived that if the same electron population produces Inverse Compton and synchrotron emission, than the slopes of the two spectra are the same. Therefore, when $U_r(\nu)$ is made by synchrotron photons, then $U_r(\nu) \propto \nu^{-\alpha}$. The result of the integral, in this case, will be $\ln(\nu_{max}/\nu_{min})$.

Fig. 4.9 helps to understand what are the right ν_{max} and ν_{min} to use. On the y–axis we have the frequencies of the seed photon distribution, which extend between ν_1 and ν_2. On the x–axis we have the scattered frequencies, which extend between $\nu_{c,1} = (4/3)\gamma_{min}^2\nu_1$ and $\nu_{c,4} = (4/3)\gamma_{max}^2\nu_1$. The diagonal lines are the functions

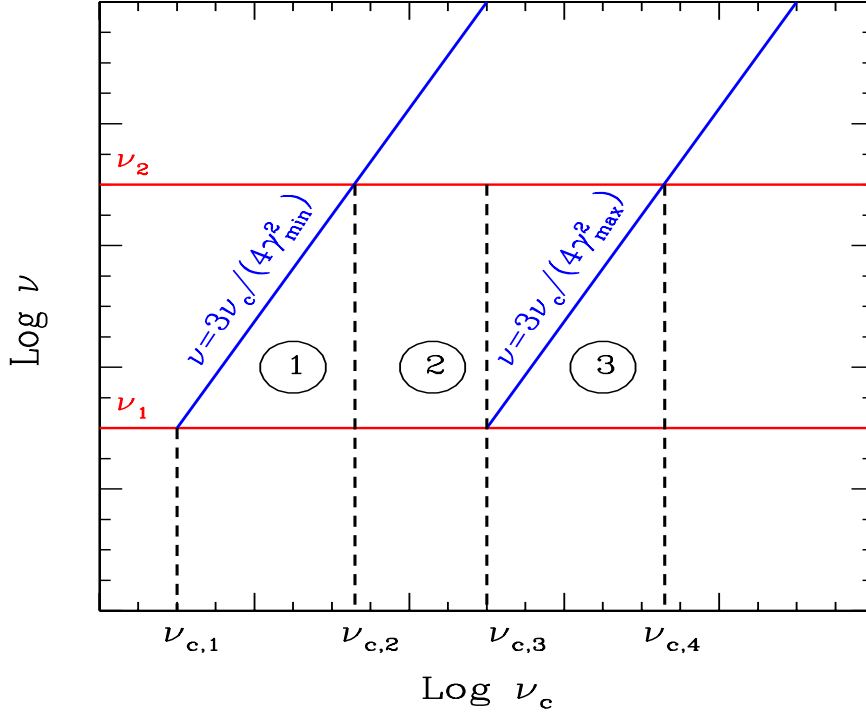$$\nu = \frac{3\nu_c}{4\gamma_{min}^2} \qquad \nu = \frac{3\nu_c}{4\gamma_{max}^2} \tag{4.53}$$

Figure 4.9: The ν–ν_c plane. The two diagonal lines delimit the regions of the seed photons that can be used to give a given frequency ν_c.

that tell us what are the appropriate ν that can give ν_c once we change γ.

There are three zones:

1. In zone (1), between $\nu_{c,1}$ and $\nu_{c,2} = (4/3)\gamma_{\min}^2\nu_2$ the appropriate limits of integration are:

$$\nu_{\min} = \nu_1 \qquad \nu_{\max} = \frac{3\nu_c}{4\gamma_{\min}^2} \qquad (4.54)$$

2. In zone (2), between $\nu_{c,2}$ and $\nu_{c,3} = (4/3)\gamma_{\max}^2\nu_1$ the limits are:

$$\nu_{\min} = \nu_1 \qquad \nu_{\max} = \nu_2 \qquad (4.55)$$

3. In zone (3), between $\nu_{c,3}$ and $\nu_{c,4} = (4/3)\gamma_{\max}^2\nu_2$ the limits are:

$$\nu_{\min} = \frac{3\nu_c}{4\gamma_{\max}^2} \qquad \nu_{\max} = \nu_2 \qquad (4.56)$$

We see that only in zone (2) the limits of integration coincide with the extension in frequency of the seed photon distribution, and are therefore constant. Therefore $\epsilon_c(\nu_c)$ will be a power law of slope α only in the corresponding frequency limits. Note also that for a narrow range $[\nu_1; \nu_2]$ or a narrow range in $[\gamma_{\min}; \gamma_{\max}]$ we do not have a power law, since there is no ν_c for which the limits of integrations are both constant.

4.7   Thermal Comptonization

With this term we mean the process of *multiple* scattering of a photon due to a *thermal* or *quasi–thermal* distribution of electrons. By *quasi–thermal* we mean a particle distribution that is peaked, even if it is not a perfect Maxwellian. Since the resulting spectrum, by definition, is due to the superposition of many spectra, each corresponding to a single scattering, the details of the particle distribution will be lost in the final spectrum, as long as the distribution is peaked. The "bible" for an extensive discussion about this process is Pozdnyakov, Sobol & Sunyaev (1983).

There is one fundamental parameter measuring the importance of the Inverse Compton process in general, and of multiple scatterings in particular: *the Comptonization parameter*, usually denoted with the letter y. Its definition is:

$$y = [\text{average \# of scatt.}] \times [\text{average fractional energy gain for scatt.}]$$
$$\text{(4.57)}$$

If $y > 1$ the Comptonization process is important, because the Comptonized spectrum has more energy than the spectrum of the seed photons.

4.7.1   Average number of scatterings

This can be calculated thinking that the photon, before leaving the source, experience a sort of random walk inside the source. Let us call

$$\tau_{\mathrm{T}} = \sigma_{\mathrm{T}} n R \qquad (4.58)$$

the Thomson scattering optical depth, where n is the electron density and R the size of the source. When $\tau_{\mathrm{T}} < 1$ most of the photons leave the source directly, without any scattering. When $\tau_{\mathrm{T}} > 1$ then the mean free path is $d = R/\tau_{\mathrm{T}}$ and the photon will experience, on average, τ_{T}^2 scatterings before leaving the source. Therefore the total path travel by the photon, from the time of its birth to the time it leaves the source is: the photon is born, is

$$c\Delta t = \tau_{\mathrm{T}}^2 \, \frac{R}{\tau_{\mathrm{T}}} = \tau_{\mathrm{T}} R \qquad (4.59)$$

and Δt is the corresponding elapsed time.

4.7.2   Average gain per scattering

Relativistic case

If the scattering electrons are relativistic, we have already seen that the photon energy is amplified by the factor $(4/3)\gamma^2$ (on average). Therefore

the problem is to find what is $\langle \gamma^2 \rangle$ in the case of a relativistic Maxwellian, that has the form

$$N(\gamma) \propto \gamma^2 e^{-\gamma/\Theta}; \quad \Theta \equiv \frac{kT}{m_e c^2} \tag{4.60}$$

Setting $x_0 = h\nu_0/(m_e c^2)$ we have that the average energy of the photon of initial frequency x_0 after a single scattering with electron belonging to this Maxwellian is:

$$
\begin{aligned}
\langle x_1 \rangle &= \frac{4}{3}\langle \gamma^2 \rangle = \frac{4}{3} x_0 \frac{\int_1^\infty \gamma^2 \gamma^2 e^{-\gamma/\Theta} d\gamma}{\int_1^\infty \gamma^2 e^{-\gamma/\Theta} d\gamma} \\
&= \frac{4}{3} x_0 \Theta^2 \frac{\Gamma(5)}{\Gamma(3)} \\
&= \frac{4}{3} x_0 \frac{4!}{2!} = 16\Theta^2 x_0
\end{aligned} \tag{4.61}
$$

Non relativistic case

In this case the average gain is proportional to the electron energy, not to its square. The derivation is not immediate, but we must use a trick. Also, we have to account that in any Maxwellian, but especially when the temperature is not large, there will be electrons that have *less* energy than the incoming photons. In this case it is the photon to give energy to the electron: correspondingly, the scattered photon will have less energy than the incoming one. Averaging out over a Maxwellian distribution, we will have:

$$\frac{\Delta x}{x} = \frac{x_1 - x_0}{x_0} = \alpha\Theta - x \tag{4.62}$$

Where $\alpha\Theta$ is what the photon gains and the $-x$ term corresponds to the downscattering of the photon (i.e. direct Compton). We do not know yet the value for the constant α. To determine it we use the following argument. We know (from general and robust arguments) what happens when photons and electrons are in equilibrium under the only process of scattering, and neglecting absorption (i.e. when the *number of photon is conserved*). What happens is that the photons follow the so–called *Wien distribution* given by:

$$F_W(x) \propto x^3 e^{-x/\Theta} \rightarrow N_W(x) = \frac{F_W(x)}{x} \propto x^2 e^{-x/\Theta} \tag{4.63}$$

where F correspond to the radiation spectrum, N to the photon spectrum, and Θ is the dimensionless electron temperature. When a Wien distribution is established we must have $\langle \Delta x \rangle = 0$, since we are at equilibrium, So we require that, on average, gains equal losses:

$$\langle \Delta x \rangle = 0 \rightarrow \alpha\Theta\langle x \rangle - \langle x^2 \rangle = 0 \tag{4.64}$$

Calculating $\langle x \rangle$ and $\langle x^2 \rangle$ for a photon Wien distribution, we have:

$$
\begin{aligned}
\langle x \rangle &= \frac{\int_0^\infty x^3 e^{-x/\Theta} dx}{\int_0^\infty x^2 e^{-x/\Theta} dx} = \frac{\Gamma(4)}{\Gamma(3)}\,\Theta = \frac{3!}{2!}\,\Theta = 3\,\Theta \\
\langle x^2 \rangle &= \frac{\int_0^\infty x^4 e^{-x/\Theta} dx}{\int_0^\infty x^3 e^{-x/\Theta} dx} = \frac{\Gamma(5)}{\Gamma(4)}\,\Theta^2 = \frac{4!}{3!}\,\Theta^2 = 12\Theta^2 \quad (4.65)
\end{aligned}
$$

This implies that $\alpha = 4$ not only at equilibrium, but always, and we finally have

$$
\frac{\Delta x}{x} = 4\Theta - x \tag{4.66}
$$

Combining the relativistic and the non relativistic cases, we have an expression valid for all temperatures:

$$
\frac{\Delta x}{x} = 16\Theta^2 + 4\Theta - x \tag{4.67}
$$

going back to the y parameter we can write:

$$
y = \max(\tau_\mathrm{T}, \tau_\mathrm{T}^2) \times [16\Theta^2 + 4\Theta - x] \tag{4.68}
$$

Going to the differential form, and neglecting downscattering, we have

$$
\frac{dx}{x} = [16\Theta^2 + 4\Theta]\,dK \;\rightarrow\; x_\mathrm{f} = x_0\,e^{(16\Theta^2 + 4\Theta)K} \;\rightarrow\; x_\mathrm{f} = x_0\,e^y \tag{4.69}
$$

where now K is the number of scatterings. If we subtract the initial photon energy, and consider that the above equation is valid for all the x_0 of the initial seed photon distribution, of luminosity L_0, we have

$$
\frac{L_\mathrm{f}}{L_0} = e^y - 1 \tag{4.70}
$$

Then the importance of y is self evident, and also the fact that it marks the importance of the Comptonization process when it is larger than 1.

4.7.3 Comptonization spectra: basics

We will illustrate why even a thermal (Maxwellian) distribution of electrons can produce a power law spectrum. The basic reason is that the total produced spectrum is the superposition of many orders of Compton scattering spectra: when they are not too much separated in frequency (i.e. for not too large temperatures) the sum is a smooth power law. We can distinguish 4 regimes, according to the values of τ_T and y. As usual, we set $x \equiv h\nu/(m_\mathrm{e}c^2)$ and $\Theta \equiv kT/(m_\mathrm{e}c^2)$.

# scatt.	Fraction of escaping photons	$\langle x \rangle$
0	$e^{-\tau_{\mathrm{T}}} \to 1 - \tau_{\mathrm{T}}$	x_0
1	$\sim \tau_{\mathrm{T}}$	$x_0 A$
2	$\sim \tau_{\mathrm{T}}^2$	$x_0 A^2$
3	$\sim \tau_{\mathrm{T}}^3$	$x_0 A^3$
4	$\sim \tau_{\mathrm{T}}^4$	$x_0 A^4$
....	......	....
n	$\sim \tau_{\mathrm{T}}^n$	$x_0 A^n$

Table 4.1: When $\tau_{\mathrm{T}} < 1$, a fraction $e^{-\tau_{\mathrm{T}}}$ of the seed photons escape without doing any scattering, and a fraction $1 - e^{-\tau_{\mathrm{T}}} \to \tau$ undergoes at least one scattering. We can then repeat these fractions for all scattering orders. Even if a tiny fraction of photons does several scatterings, they can carry a lot of energy.

The case $\tau_{\mathrm{T}} < 1$

Neglect downscattering for simplicity. The fractional energy gain is $\Delta x / x = 16\Theta^2 + 4\Theta$, so the amplification A of the photon frequency at each scattering is

$$A \equiv \frac{x_1}{x} = 16\Theta^2 + 4\Theta + 1 \sim \frac{y}{\tau_{\mathrm{T}}} \tag{4.71}$$

We can then construct the following table

A look to Fig. 4.10 should convince you that the sum of all the scattering orders gives a power law, and should also make clear how to find the spectral slope. Remember that we are in a log–log plot, so the spectral index is simply $\Delta y / \Delta x$. We can find it considering two successive scattering orders: the typical (logarithm of) frequency is enhanced by $\log A$, and the fraction of photons doing the scattering is $-\log \tau_{\mathrm{T}}$. Remember also that we use $F(x) \propto x^{-\alpha}$ as the definition of energy spectral index.

Therefore

$$\alpha = -\frac{\log \tau_{\mathrm{T}}}{\log A} \sim -\frac{\log \tau_{\mathrm{T}}}{\log y - \log \tau_{\mathrm{T}}} \tag{4.72}$$

When $y \sim 1$, its logarithm is close to zero, and we have $\alpha \sim 1$. When $y > 1$, then $\alpha > 1$ (i.e. *flat, or hard*), and vice–versa, when $y < 1$, then its logarithm will be negative, as the logarithm of τ_{T}, and $\alpha > 1$ (i.e. *steep, or soft*). Therefore

$$\alpha = -\frac{\log \tau_{\mathrm{T}}}{\log A} \sim -\frac{\log \tau_{\mathrm{T}}}{\log y - \log \tau_{\mathrm{T}}} \tag{4.73}$$

When $y \sim 1$, its logarithm is close to zero, and we have $\alpha \sim 1$. When $y > 1$, then $\alpha > 1$ (i.e. *flat, or hard*), and vice–versa, when $y < 1$, then its logarithm will be negative, as the logarithm of τ_{T}, and $\alpha > 1$ (i.e. *steep, or soft*).
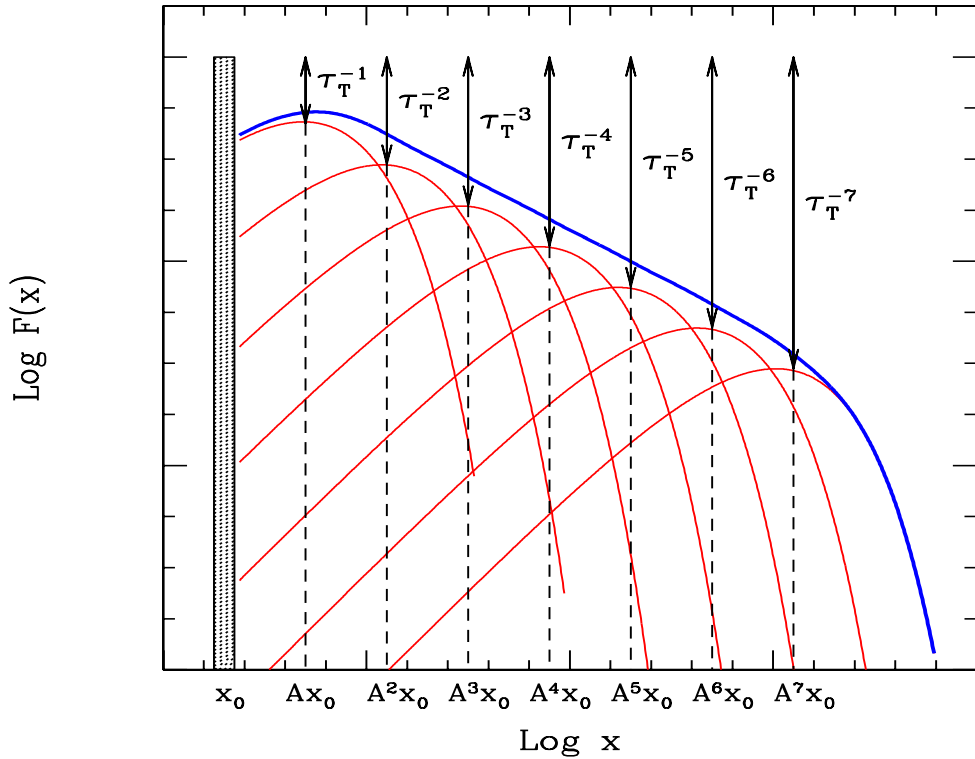
Figure 4.10: Multiple Compton scatterings when $\tau_{\mathrm{T}} < 1$. A fraction τ_{T} of the photons of the previous scattering order undergoes another scattering, and amplifying the frequency by the gain factor A, until the typical photon frequency equals the electron temperature Θ. Then further scatterings leave the photon frequency unchanged.

Attention! when $\tau_{\mathrm{T}} \ll 1$ and A is large (i.e. big frequency jumps between one scattering and the next), then the superposition of all scattering orders (by the way, there are fewer, in this case) will not guarantee a perfect power–law. In the total spectrum we can see the "bumps" corresponding to individual scattering orders.

The case $\tau_{\mathrm{T}} \gtrsim 1$

This is the most difficult case, as we should solve a famous equation, the equation of Kompaneet. The result is still a power law, whose spectral index is approximately given by

$$\alpha = -\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{4}{y}} \tag{4.74}$$

The case $\tau_{\mathrm{T}} \gg 1$: saturation

In this case the interaction between photons and matters is so intense that they go to equilibrium, and they will have the same temperature. But instead of a black–body, the resulting photon spectrum has a Wien shape. This is because the photons are conserved (and therefore, if other scattering processes such as induced Compton or two–photon scattering are important, then one recovers a black–body, because these processes *do not* conserve photons). The Wien spectrum has the slope:

$$I(x) \propto x^3 \, e^{-x/\Theta} \tag{4.75}$$

At low frequencies this is *harder* than a black–body.

The case $\tau_{\mathrm{T}} > 1$, $y > 1$: quasi–saturation

Suppose that in a source characterized by a large τ_{T} the source of soft photons is spread throughout the source. In this case the photons produced close to the surface, in a skin of optical depth $\tau_{\mathrm{T}} = 1$, leave the source without doing any scattering (note that having the source of seed photons concentrated at the center is a different case). The remaining fraction, $1 - 1/\tau_{\mathrm{T}}$, i.e. *almost* all photons, remains inside. This can be said for each scattering order. This is illustrated in Fig. 4.11, where τ_{T} corresponds to the ratio between the flux of photon inside the source at a given frequency and the flux of photons that escape. If I start with 100 photons, only 1 – say – escape, and the other 99 remain inside, and do the first scattering. After it, only one escape, and the other 98 remain inside, and so on, until the typical photon and electron energies are equal, and the photon therefore stays around with the same final frequency until it is its turn to escape. This "accumulation" of photons at $x \sim 3\Theta$ gives the Wien bump. Note that since at any scattering order only a fixed number of photons escape, always the same, then the spectrum in this region will always have $\alpha = 0$. This is a "saturated" index, i.e. one obtains always zero even when changing τ_{T} or Θ. What indeed changes, by increasing τ_{T}, is that i) the flux characterized by x^0 decreases, ii) the Wien peak will start to dominate earlier (at lower frequencies), while nothing happens to the flux of the Wien peak (it stays there). Increasing still τ_{T} we fall in the previous case (equilibrium, meaning only the Wien spectrum without the x^0 part).
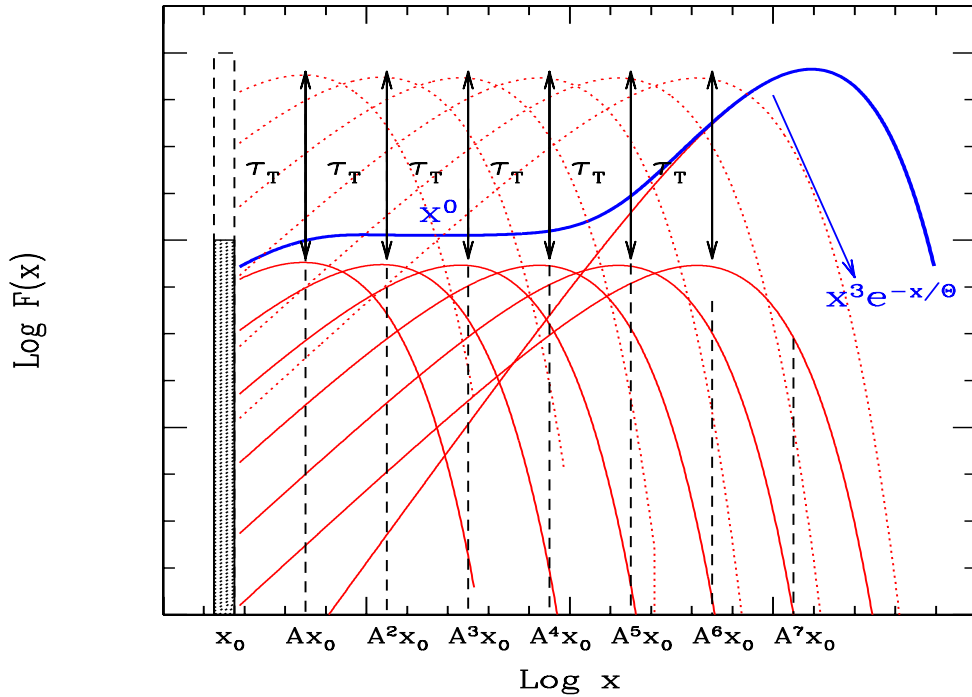
Figure 4.11: Multiple Compton scatterings when $\tau_T > 1$ and $y \gg 1$. For the first scattering orders, *nearly all* photons are scattered: only a fraction $1/\tau_T$ can escape. Therefore the number of photons escaping at each scattering order is the same. This is the reason of the flat part, where $F(x) \propto x^0$. When the photon frequency is of the order of Θ, photons and electrons are in equilibrium, and even if only a small fraction of photon can escape at each scattering order, they do not change frequency any longer, and therefore they form the *Wien bump*, with the slope $F(x) \propto x^3 e^{-x/\Theta}$. If we increase τ_T, the flux with slope x^0 decreases, while the Wien bump stays the same.

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Chapter 5

Synchrotron Self–Compton

Consider a population of relativistic electrons in a magnetized region. They will produce synchrotron radiation, and therefore they will fill the region with photons. These synchrotron photons will have some probability to interact again with the electrons, by the Inverse Compton process. Since the electron "work twice" (first making synchrotron radiation, then scattering it at higher energies) this particular kind of process is called synchrotron self–Compton, or SSC for short.

5.1 SSC emissivity

The importance of the scattering will of course be high if the densities of electrons and photons are large. If the electron distribution is a power law $[N(\gamma) = K\gamma^{-p}]$, then we expect that the SSC flux will be $\propto K^2$, i.e. *quadratic* in the electron density.

We should remember Eq. 4.52, and, instead of a generic $U_\mathrm{r}(\nu)$, we should substitute the appropriate expression for the specific synchrotron radiation energy density. We will then set:

$$U_\mathrm{s}(\nu) = \frac{3R}{4c}\frac{L_\mathrm{s}(\nu)}{V} = 4\pi\,\frac{3R}{4c}\,\epsilon_\mathrm{s}(\nu) \qquad (5.1)$$

where $3R/(4c)$ is the average photon source–crossing time, and V is the volume of the source. Now a simple trick: we write the specific synchrotron emissivity as

$$\epsilon_\mathrm{s}(\nu) = \epsilon_{\mathrm{s},0}\,\nu^{-\alpha} \qquad (5.2)$$

Remember: the α appearing here *is the same* index in Eq. 4.52. Substituting the above equations into Eq. 4.52 we have

$$\epsilon_\mathrm{ssc}(\nu_\mathrm{c}) = \frac{(4/3)^{\alpha-1}}{2}\,\tau_\mathrm{c}\,\epsilon_{\mathrm{s},0}\nu_\mathrm{c}^{-\alpha}\int_{\nu_\mathrm{min}}^{\nu_\mathrm{max}}\frac{d\nu}{\nu} \qquad (5.3)$$

As you can see, $\nu_{\rm c}^{-\alpha}$ is nothing else than the specific synchrotron emissivity calculated at the (Compton) frequency $\nu_{\rm c}$. Furthermore, the integral gives a logarithmic term, that we will call $\ln \Lambda$. We finally have:

$$\epsilon_{\rm ssc}(\nu_{\rm c}) \,=\, \frac{(4/3)^{\alpha-1}}{2}\, \tau_{\rm c}\, \epsilon_{\rm s}(\nu_{\rm c}) \ln \Lambda \qquad (5.4)$$

In this form the ratio between the synchrotron and the SSC flux is clear, it is $[(4/3)^{\alpha-1}/2]\tau_{\rm c} \ln \Lambda \sim \tau_{\rm c} \ln \Lambda$. It is also clear that since $\tau_{\rm c} \equiv \sigma_{\rm T} R K$ and $\epsilon_{\rm s}(\nu_{\rm c}) \propto K B^{1+\alpha}$, then, as we have guessed, the SSC emissivity $\epsilon_{\rm ssc}(\nu_{\rm c}) \propto K^2$ (i.e. electrons work twice). Fig. 5.1 summarizes the main results.
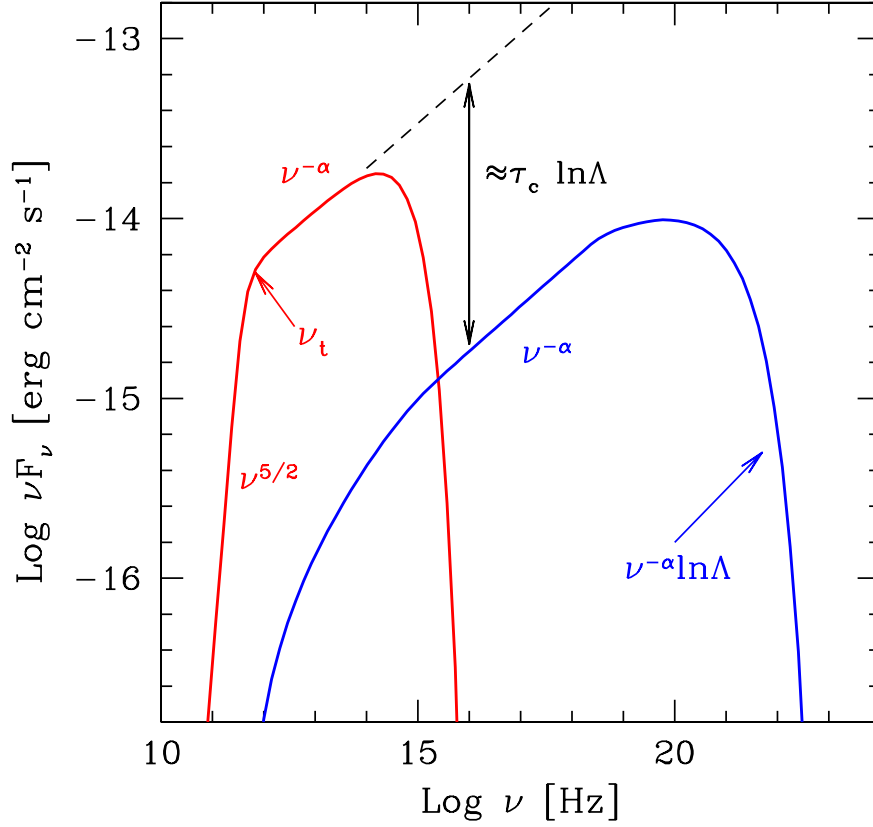

Figure 5.1: Typical example of SSC spectrum, shown in the νF_ν vs ν representation. The spectral indices instead correspond to the $F_\nu \propto \nu^{-\alpha}$ convention.

5.2 Diagnostic

If we are confident that a the spectrum of a particular source is indeed given by the SSC process, then we can use our theory to estimate a number of physical parameters. We have already stated (see Eq. 2.32) that observations of the synchrotron spectrum in its self–absorbed part can yield the value of the magnetic field if we also know the angular radius of the source (if it is resolved). Observation in the thin part can then give us the product $RK \equiv \tau_c/\sigma_T$ (see Eq. 2.28). But τ_c is exactly what we need to predict the high energy flux produced by the SSC process. Note that if the source is resolved (i.e. we know θ_s) we can get these information even without knowing the distance of the source. To summarize:

$$
\begin{aligned}
F^{\mathrm{syn}}_{\mathrm{thick}}(\nu) &\propto \theta_s^2 \frac{\nu^{5/2}}{B^{1/2}} &\rightarrow& \ \ \text{get } B \\
F^{\mathrm{syn}}_{\mathrm{thin}}(\nu) &\propto \theta_s^2 RKB^{1+\alpha}\nu^{-\alpha} &\rightarrow& \ \text{get } \tau_c = RK/\sigma_T
\end{aligned}
\tag{5.5}
$$

There is an even simpler case, which for reasons outlined below, is the most common case employed when studying radio–loud AGNs. In fact, if you imagine to observe the source at the self absorption frequency ν_t, then you are both observing the thick and the thin flux at the same time. Then, let us call the flux at ν_t simply F_t. We can then re-write the equation above:

$$
\begin{aligned}
B &\propto \frac{\theta_s^4 \nu_t^5}{F_t^2} \\
\tau_c &\propto \frac{F_t \nu_t^\alpha}{\theta_s^2 B^{1+\alpha}} \\
F_{\mathrm{ssc}}(\nu_c) &\propto \tau_c F_{syn}(\nu_c) \propto \tau_c^2 B^{1+\alpha}\nu_c^{-\alpha} \\
&\propto F_t^{2(2+\alpha)}\nu_t^{-(5+3\alpha)}\theta_s^{-2(3+2\alpha)}\nu_c^{-\alpha}
\end{aligned}
\tag{5.6}
$$

Once again: on the basis of a few observations of only the synchrotron flux, we can calculate what should be the SSC flux at the frequency ν_c. Note the rather strong dependencies, particularly for θ_s, in the sense that the more compact the source is, the larger the SSC flux.

If it happens that we do observe the source at high frequencies, where we expect that the SSC flux dominates, then we can check if our model works. Does it? For the strongest radio–loud sources, almost never. The disagreement between the predicted and the observed flux is really severe, we are talking of several orders of magnitude. Then either we are completely wrong about the model, or we miss some fundamental ingredient. We go for the second option, since, after all, we do not find any mistake in our theory.

The missing ingredient is relativistic bulk motion. If the source is moving towards us at relativistic velocities, we observe an enhanced flux and blueshifted frequencies. Not accounting for it, our estimates of the magnetic

field and particles densities are wrong, in the sense that the B field is smaller than the real one, and the particle densities are much greater (for smaller B we need more particle to produce the same synchrotron flux). So we repeat the entire procedure, but this time assuming that $F(\nu) = \delta^{3+\alpha} F'(\nu)$, where $\delta = 1/[\Gamma(1 - \beta \cos\theta)]$ is the Doppler factor and $F'(\nu)$ is the flux received by a comoving observer at the same frequency ν. Then

$$
\begin{aligned}
F_{\text{thick}}^{\text{syn}}(\nu) &\propto \theta_s^2 \frac{\nu^{5/2}}{B^{1/2}} \delta^{1/2} \\
F_{\text{thin}}^{\text{syn}}(\nu) &\propto \theta_s^2 RKB^{1+\alpha} \nu^{-\alpha} \delta^{3+\alpha}
\end{aligned}
\tag{5.7}
$$

The predicted SSC flux then becomes

$$
F_{\text{ssc}}(\nu_{\text{c}}) \propto F_{\text{t}}^{2(2+\alpha)} \nu_{\text{t}}^{-(5+3\alpha)} \theta_s^{-2(3+2\alpha)} \nu_{\text{c}}^{-\alpha} \delta^{-2(2+\alpha)}
\tag{5.8}
$$

If we now compare the predicted with the observe SSC flux, we can estimate δ. And indeed this is one of the most powerful δ–estimators, even if it is not the only one.

5.3   Why it works

We have insisted on the importance of observing the synchrotron flux both in the self–absorbed and in the thin regime, to get B and τ_{c}. But the self–absorbed part of the synchrotron spectrum, the one $\propto \nu^{5/2}$ is very rarely observed in general, and never in radio–loud AGNs. So, where is the trick? It is the following. In radio–loud AGN the synchrotron emission, at radio frequencies, comes partly from the radio lobes (extended structures, hundreds of kpc in size, very relaxed, unbeamed, and usually self–absorbing at very small frequencies) and from the jet. The emission from the latter is beamed, and it is the superposition of the fluxes produced in several regions: the most compact ones (closer to the central engine) self–absorb at high radio frequencies (say, at 100 GHz), and the bigger they are, the smaller their self–absorbed frequency. But what is extraordinary about these jets is that the peak flux of each component (i.e. the flux at the self–absorption frequency) is approximately constant (in the past, this phenomenon was called *cosmic conspiracy*). Therefore, when we sum up all the components, we have a flat radio spectrum, as illustrated by Fig. 5.2.

Of course the emission components of the jet, to behave in such a coherent way, must have an electron density and a magnetic field that decrease with the distance from the central engine in an appropriate way. There is a family of solutions, but the most appealing is certainly $B(R) \propto R^{-1}$ and $K(R) \propto R^{-2}$. It is appealing because it corresponds to conservation of the total number of particles, conservation of the bulk power carried by them (if Γ does not change) and conservation of the Poynting flux (i.e. the power carried by the magnetic field).
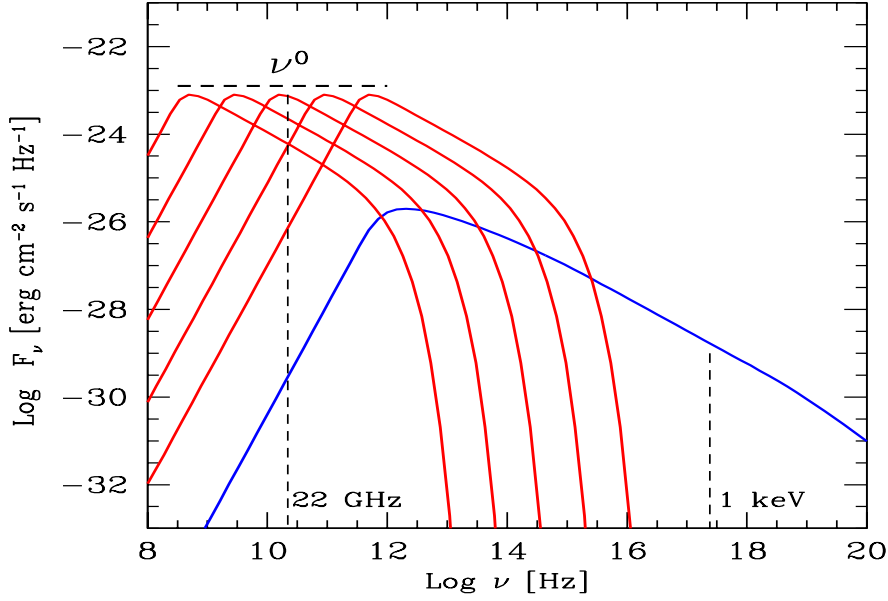
Figure 5.2: Typical example of the composite spectrum of a flat spectrum quasars (FSRQ) shown in the F_ν vs ν representation, to better see the flat spectrum in the radio. The reason of the flat spectrum is that different parts of the jet contributes at different frequencies, but in a coherent way. The blue line is the SSC spectrum. Suppose to observe, with the VLBI, at 22 GHz: in this framework we will always observe the jet component peaking at this frequency. So you automatically observe at the self–absorption frequency of that component (for which you measure the angular size).

To our aims, the fact that the jet has many radio emission sites self–absorbing at different frequency is of great help. In fact suppose to observe a jet with the VLBI, at one frequency, say 22 GHz. There is a great chance to observe the emission zone which is contributing the most to that frequency, i.e. the one which is self–absorbing at 22 GHz. At the same time you measure the size. Then, suppose to know the X–ray flux of the source. It will be the X–ray flux not only of that component you see with the VLBI, but an integrated flux from all the inner jet (with X–ray instruments the maximum angular resolution is about 1 arcsec, as in optical). But nevertheless you know that your radio blob cannot exceed the measured, total, X–ray flux. Therefore you can put a limit on δ (including constants):

$$
\begin{aligned}
\delta \quad > \quad & (0.08\alpha + 0.14)\,(1 + z) \left(\frac{F_{\rm t}}{\rm Jy}\right)\left(\frac{F_x}{\rm Jy}\right)^{-\frac{1}{2(2+\alpha)}}\left(\frac{\nu_x}{1\ {\rm keV}}\right)^{-\frac{\alpha}{2(2+\alpha)}} \\
\times \quad & \left(\frac{\nu_{\rm t}}{5\ {\rm GHz}}\right)^{-\frac{5+3\alpha}{2(2+\alpha)}}\left(\frac{2\theta_s}{\rm m.a.s.}\right)^{-\frac{3+2\alpha}{2+\alpha}}\left[\ln\left(\frac{\nu_{\rm s,max}}{\nu_{\rm t}}\right)\right]^{\frac{1}{2(2+\alpha)}} \quad\quad (5.9)
\end{aligned}
$$

For some sources you would find $\delta > 10$ or 20, i.e. rather large values.